Master Degree (LM) in Electronic Engineering

**MICROWAVES**

**MICROWAVE RESONATORS**

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Chapter 6
SUMMARY

• Series and Parallel Resonant Circuits
• Loaded and Unloaded quality factor Q
• Transmission Line Resonators
• Rectangular Waveguide Cavity Resonators
• Circular Waveguide Cavity Resonators
• Dielectric Resonators
• Excitation of Resonators
• Cavity Perturbations
Microwave resonators are used in a variety of applications, including filters, oscillators, frequency meters, and tuned amplifiers.

The operation of microwave resonators is very similar to that of lumped-element resonators of circuit theory, therefore the basic characteristics of series and parallel RLC resonant circuits will be revised.

Different implementations of resonators at microwave frequencies will be discussed, using distributed elements such as transmission lines, rectangular and circular waveguides, and dielectric cavities.

Finally, the excitation of resonators using apertures and current sheets will be addressed.
**SERIES RESONANT CIRCUIT**

The input impedance is

\[ Z_{\text{in}} = R + j\omega L - j \frac{1}{\omega C} \]

and the complex power delivered to the load is

\[ P_{\text{in}} = \frac{1}{2} VI^* = \frac{1}{2} Z_{\text{in}}|I|^2 = \frac{1}{2} Z_{\text{in}} \left| \frac{V}{Z_{\text{in}}} \right|^2 = \frac{1}{2} |I|^2 \left( R + j\omega L - j \frac{1}{\omega C} \right) \]
SERIES RESONANT CIRCUIT

The active power lost in the RLC and the reactive powers of the inductor (magnetic) and of the capacitor (electric) are

\[ P_{\text{loss}} = \frac{1}{2}|I|^2 R \quad W_m = \frac{1}{4}|I|^2 L \quad W_e = \frac{1}{4}|V_c|^2 C = \frac{1}{4}|I|^2 \frac{1}{\omega^2 C} \]

Therefore, the complex power and the input impedance can be written as

\[ P_{\text{in}} = P_{\text{loss}} + 2j\omega(W_m - W_e) \]

\[ Z_{\text{in}} = \frac{2P_{\text{in}}}{|I|^2} = \frac{P_{\text{loss}} + 2j\omega(W_m - W_e)}{\frac{1}{2}|I|^2} \]

Resonator losses represented by \( R \) may be due to conductor loss, dielectric loss, or radiation loss.
Resonance occurs when the average stored magnetic and electric energies are equal \((W_m = W_e)\), and the input impedance at resonance become

\[
Z_{\text{in}} = \frac{P_{\text{loss}}}{\frac{1}{2}|I|^2} = R \quad \text{(purely real)}
\]

From \(W_m = W_e\), the resonant angular frequency \(\omega_0 = 2\pi f_0\) results

\[
\omega_0 = \frac{1}{\sqrt{LC}}
\]
Another important parameter of a resonant circuit is the quality factor $Q$, defined as

$$Q = \omega \frac{\text{average energy stored}}{\text{energy loss/second}} = \omega \frac{W_m + W_e}{P_{\text{loss}}}$$

The quality factor is a measure of the loss of a resonant circuit (lower loss implies higher $Q$). If there is no interaction with an external network, this is called the unloaded $Q$.

For the series resonant circuit the unloaded $Q$ can be evaluated as

$$Q_0 = \omega_0 \frac{2W_m}{P_{\text{loss}}} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$
Since $\omega_0^2 = 1/LC$, the input impedance can be written as

$$Z_{in} = R + j\omega L \left(1 - \frac{1}{\omega^2LC}\right) = R + j\omega L \left(\frac{\omega^2 - \omega_0^2}{\omega^2}\right)$$

Near the resonant frequency, i.e. for $\omega = \omega_0 + \Delta\omega$ with $\Delta\omega \ll \omega_0$, we have

$$\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) = \Delta\omega(2\omega - \Delta\omega) \simeq 2\omega\Delta\omega$$

and

$$Z_{in} \simeq R + j2L\Delta\omega \simeq R + j\frac{2RQ_0\Delta\omega}{\omega_0}$$
The last formula for $Z_{\text{in}}$ can also be used to infer how to model a lossy high-Q resonator by perturbation of his lossless counterpart.

The input impedance of a lossless series LC circuit is

$$Z_{\text{in}} = j2L\Delta\omega = j2L(\omega - \omega_0)$$

By substituting

$$\omega_0 \left(1 + \frac{j}{2Q_0}\right) \to \omega_0$$

and remembering $Q_0 = \omega_0 L/R$ it results

$$Z_{\text{in}} = j2L\Delta\omega = j2L \left[\omega - \omega_0 \left(1 + \frac{j}{2Q_0}\right)\right] = \frac{\omega_0 L}{Q_0} + j2L(\omega - \omega_0) = R + j2L(\omega - \omega_0)$$

This procedure is general and applies to all the low-losses resonators.
SERIES RESONANT CIRCUIT

The half-power fractional bandwidth BW of the resonator is defined as the percentage frequency bandwidth across $\omega_0$ with the active power delivered not less that one half of the maximum.

Since the active power delivered to the resonator is

$$P_a = Re \left\{ Z_{in} \left| \frac{V}{Z_{in}} \right|^2 \right\} = R \frac{|V|^2}{|Z_{in}|^2}$$

the maximum active power results

$$P_{a,\text{max}} = R \frac{|V|^2}{R^2}$$

Therefore, since $\Delta \omega / \omega_0 = BW / 2$, the power is one half when

$$|Z_{in}|^2 = |R + jRQ_0(BW)|^2 = 2R^2$$

from which

$$BW = \frac{1}{Q_0}$$
In the case of a parallel RLC circuit, the results are similar.

In particular, the quality factor is

$$Q_0 = \omega_0 \frac{2W_m}{P_{loss}} = \frac{R}{\omega_0 L} = \omega_0 RC$$

and the input impedance close to the resonance results

$$Z_{in} \simeq \frac{R}{1 + 2j\Delta\omega RC} = \frac{R}{1 + 2jQ_0\Delta\omega/\omega_0}$$

Also in this case the half-power fractional bandwidth $BW$ is

$$BW = \frac{1}{Q_0}$$

See the Pozar's book for more details and for the derivation.
LOADED AND UNLOADED $Q$

An external connecting network may introduce additional loss. Each of these loss mechanisms will have the effect of lowering the $Q$.

For a series RLC, the effective resistance become $R + R_L$.

For a parallel RLC, the effective resistance become $RR_L/(R + R_L)$.

If we define an external quality factor $Q_e$ as

$$Q_e = \begin{cases} \frac{\omega_0 L}{R_L} & \text{for series circuits} \\ \frac{R_L}{\omega_0 L} & \text{for parallel circuits} \end{cases}$$

then the loaded $Q_L$ can be expressed as

$$\frac{1}{Q_L} = \frac{1}{Q_e} + \frac{1}{Q_0}$$
## Loaded and Unloaded $Q$

### In summary

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Series Resonator</th>
<th>Parallel Resonator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input impedance/admittance</td>
<td>$Z_{in} = R + j\omega L - j\frac{1}{\omega C}$</td>
<td>$Y_{in} = \frac{1}{R} + j\omega C - j\frac{1}{\omega L}$</td>
</tr>
<tr>
<td></td>
<td>$\simeq R + j\frac{2RQ_{0}\Delta\omega}{\omega_{0}}$</td>
<td>$\simeq \frac{1}{R} + j\frac{2Q_{0}\Delta\omega}{R\omega_{0}}$</td>
</tr>
<tr>
<td>Power loss</td>
<td>$P_{loss} = \frac{1}{2}</td>
<td>I</td>
</tr>
<tr>
<td>Stored magnetic energy</td>
<td>$W_{m} = \frac{1}{4}</td>
<td>I</td>
</tr>
<tr>
<td>Stored electric energy</td>
<td>$W_{e} = \frac{1}{4}</td>
<td>I</td>
</tr>
<tr>
<td>Resonant frequency</td>
<td>$\omega_{0} = \frac{1}{\sqrt{LC}}$</td>
<td>$\omega_{0} = \frac{1}{\sqrt{LC}}$</td>
</tr>
<tr>
<td>Unloaded $Q$</td>
<td>$Q_{0} = \frac{\omega_{0}L}{R} = \frac{1}{\omega_{0}RC}$</td>
<td>$Q_{0} = \omega_{0}RC = \frac{R}{\omega_{0}L}$</td>
</tr>
<tr>
<td>External $Q$</td>
<td>$Q_{e} = \frac{\omega_{0}L}{R_{L}}$</td>
<td>$Q_{e} = \frac{R_{L}}{\omega_{0}L}$</td>
</tr>
</tbody>
</table>
Open/short transmission lines of special length (half wavelength or quarter wavelength) can be used as resonators when lumped element circuits are unfeasible due to the high frequency.

To study the quality factor, lossy transmission lines will be considered.
**TRANSMISSION LINE RESONATORS**

Short-circuited $\lambda/2$ line

At the resonant frequency $\omega = \omega_0$, the length of the line is $\ell = \lambda/2$.

The input impedance is

$$Z_{in} = Z_0 \tanh(\alpha + j\beta)\ell$$

$$= Z_0 \frac{\tanh \alpha \ell + j \tan \beta \ell}{1 + j \tan \beta \ell \tanh \alpha \ell}$$

If we are interested in a small frequency band across $\omega_0$, we can set $\omega = \omega_0 + \Delta \omega$ ($\Delta \omega \ll \omega_0$).

Moreover, if the mode propagating is TEM ($\ell = \frac{\lambda}{2} = \frac{\pi v_p}{\omega_0}$) and $\alpha \ell \ll 1$, we have

$$\tan \beta \ell = \tan \left( \frac{\omega_0 + \Delta \omega}{v_p} \ell \right) = \tan \left( \pi + \frac{\Delta \omega}{\omega_0} \pi \right) \approx \frac{\Delta \omega}{\omega_0} \pi$$

$$\tanh \alpha \ell \approx \alpha \ell$$
Short-circuited $\lambda/2$ line

By substituting and observing that $\Delta \omega \alpha \ell / \omega_0 \ll 1$, the approximated input impedance results

$$Z_{\text{in}} \approx Z_0 \frac{\alpha \ell + j (\Delta \omega \pi / \omega_0)}{1 + j (\Delta \omega \pi / \omega_0) \alpha \ell} \approx Z_0 \left( \alpha \ell + j \frac{\Delta \omega \pi}{\omega_0} \right)$$

It can be cast in the form,

$$Z_{\text{in}} = R + 2 j L \Delta \omega,$$

which is the same expression of the series RLC circuit.

The lumped element values are

$$R = Z_0 \alpha \ell \quad L = \frac{Z_0 \pi}{2 \omega_0} \quad C = \frac{1}{\omega_0^2 L}$$

The quality factor is

$$Q_0 = \frac{\omega_0 L}{R} = \frac{\pi}{2 \alpha \ell} = \frac{\beta}{2 \alpha}$$
**TRANSMISSION LINE RESONATORS**

Short-circuited $\lambda/4$ line

At the resonant frequency $\omega = \omega_0$, the length of the line is $\ell = \lambda/4$.

The input impedance is

$$Z_{in} = Z_0 \tanh(\alpha + j\beta)\ell$$

$$= Z_0 \frac{\tanh \alpha \ell + j \tan \beta \ell}{1 + j \tan \beta \ell \tanh \alpha \ell}$$

$$= Z_0 \frac{1 - j \tanh \alpha \ell \cot \beta \ell}{\tanh \alpha \ell - j \cot \beta \ell}$$

We are interested in a small frequency band across $\omega_0$: $\omega = \omega_0 + \Delta \omega$ ($\Delta \omega \ll \omega_0$).

Moreover, if the mode propagating is TEM ($\ell = \frac{\lambda}{4} = \frac{\pi v_p}{2\omega_0}$) and $\alpha \ell \ll 1$, we have

$$\cot \beta \ell = \cot \left(\frac{\pi}{2} + \frac{\pi \Delta \omega}{2 \omega_0}\right) = -\tan \left(\frac{\pi \Delta \omega}{2 \omega_0}\right) \approx -\frac{\pi \Delta \omega}{2 \omega_0}$$

$$\tanh \alpha \ell \approx \alpha \ell$$
TRANSMISSION LINE RESONATORS

Short-circuited $\lambda/4$ line

By substituting and observing that $\Delta \omega \alpha \ell / (2 \omega_0) \ll 1$, the approximated input impedance results

$$Z_{in} = Z_0 \frac{1 + j \alpha \ell \pi \Delta \omega / 2 \omega_0}{\alpha \ell + j \pi \Delta \omega / 2 \omega_0} \approx \frac{Z_0}{\alpha \ell + j \pi \Delta \omega / 2 \omega_0}$$

It can be cast in the form,

$$Z_{in} = \frac{1}{(1/R) + 2j \Delta \omega C}$$

which is the same expression of the parallel RLC circuit.

The lumped element values are

$$R = \frac{Z_0}{\alpha \ell} \quad L = \frac{1}{\omega_0^2 C} \quad C = \frac{\pi}{4 \omega_0 Z_0}$$

The quality factor is

$$Q_0 = \omega_0 RC = \frac{\pi}{4 \alpha \ell} = \frac{\beta}{2 \alpha}$$
Open-circuited $\lambda/2$ line

At the resonant frequency $\omega = \omega_0$, the length of the line is $\ell = \lambda/2$.

The input impedance is

$$Z_{\text{in}} = Z_0 \coth(\alpha + j\beta)\ell$$

$$= Z_0 \frac{1 + j\tan\beta\ell \tanh\alpha\ell}{\tanh\alpha\ell + j\tan\beta\ell}$$

We are interested in a small frequency band across $\omega_0$: $\omega = \omega_0 + \Delta\omega$ ($\Delta\omega \ll \omega_0$).

Moreover, if the mode propagating is TEM ($\ell = \frac{\lambda}{2} = \frac{\pi v_p}{\omega_0}$) and $\alpha\ell \ll 1$, we have

$$\tan\beta\ell = \tan\left(\pi + \frac{\Delta\omega}{\omega_0}\pi\right) \approx \frac{\Delta\omega}{\omega_0}\pi$$

$$\tanh\alpha\ell \approx \alpha\ell$$
Open-circuited $\lambda/2$ line

By substituting and observing that $\Delta \omega \alpha \ell / \omega_0 \ll 1$, the approximated input impedance results

$$Z_{in} = \frac{Z_0}{\alpha \ell + j (\Delta\omega \pi / \omega_0)}$$

It can be cast in the form,

$$Z_{in} = \frac{1}{(1/R) + 2j\Delta\omega C}$$

which is the same expression of the parallel RLC circuit.

The lumped element values are

$$R = \frac{Z_0}{\alpha \ell} \quad L = \frac{1}{\omega_0^2 C} \quad C = \frac{\pi}{2\omega_0 Z_0}$$

The quality factor is

$$Q_0 = \omega_0 RC = \frac{\pi}{2\alpha \ell} = \frac{\beta}{2\alpha}$$
Let’s consider a resonator comprising many sections of transmission lines with different characteristic impedances $Z_1$, $Z_2$, ..., $Z_N$, loaded at the extremes with arbitrary reactive impedances $jX_a$ and $jX_b$:

The characteristic equation to determine the resonant frequencies can be obtained by cutting the structure in any section and imposing the condition

$$Z_L = Z_R^*$$
**RECTANGULAR WAVEGUIDE CAVITY RESONATORS**

Waveguide cavities can be made **without dielectrics**, and therefore they are preferred when **high-Q** is required.

A cavity based on a rectangular waveguide is usually obtained by **short circuiting** the waveguide at both ends.

An infinite number of modes resonate in the cavity, and they can be derived from propagating TE and TM modes in the waveguide.

The resonant frequencies and field expressions will be derived under the hypothesis of a **lossless resonator**.

The **Q factor** will be derived by using a perturbation technique.
RECTANGULAR WAVEGUIDE CAVITY RESONATORS

It is sufficient to impose the perfect electric wall condition at $z = 0, d$ to the field of the TE or TM propagating modes

$$\tilde{E}_t(x, y, z) = \tilde{e}(x, y) \left( A^+ e^{-j \beta_{mn} z} + A^- e^{j \beta_{mn} z} \right)$$

to determine the resonant frequency and the resonant modal field.

Enforcing $\tilde{E}_t(x, y, 0) = \tilde{E}_t(x, y, d) = 0$ leads to

$$\tilde{E}_t(x, y, d) = -\tilde{e}(x, y) A^+ 2j \sin \beta_{mn} d = 0$$

and, therefore

$$\beta_{mn} d = \ell \pi, \quad \ell = 1, 2, 3, \ldots,$$

where

$$\beta_{mn} = \sqrt{k^2 - \left( \frac{m \pi}{a} \right)^2 - \left( \frac{n \pi}{b} \right)^2},$$
RECTANGULAR WAVEGUIDE CAVITY RESONATORS

The resonance wavenumber is therefore

\[ k_{mn\ell} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2} \]

and the resonance frequency results

\[ f_{mn\ell} = \frac{ck_{mn\ell}}{2\pi\sqrt{\mu_r\varepsilon_r}} = \frac{c}{2\pi\sqrt{\mu_r\varepsilon_r}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2} \]

If \( b < a < d \) the dominant resonant mode (lowest resonant frequency) will be the \( \text{TE}_{101} \) mode, corresponding to the \( \text{TE}_{10} \) dominant waveguide mode in a shorted guide of length \( \lambda_g/2 \). The dominant TM resonant mode is the \( \text{TM}_{110} \) mode.
RECTANGULAR WAVEGUIDE CAVITY RESONATORS

Unloaded Q of the TE\textsubscript{10ℓ} mode

To evaluate the Q, we need to determine the electric and magnetic stored energies, as well as the dissipated power.

Taking into account the expression of the field of the propagating TE\textsubscript{10} mode, and using the conditions deriving from the enforcement of the resonance, the resonant field results

\[
E_y = E_0 \sin \frac{\pi x}{a} \sin \frac{\ell \pi z}{d},
\]

\[
\begin{align*}
H_x &= -jE_0 \frac{1}{Z_{TE}} \sin \frac{\pi x}{a} \cos \frac{\ell \pi z}{d} \\
H_z &= \frac{j \pi E_0}{k \eta a} \cos \frac{\pi x}{a} \sin \frac{\ell \pi z}{d}
\end{align*}
\]
RECTANGULAR WAVEGUIDE CAVITY RESONATORS

Unloaded Q of the TE_{10ℓ} mode

The stored energies are derived with some simple calculations:

\[ W_e = \frac{\epsilon}{4} \int_V E_y E_y^* dV = \frac{\epsilon ab d}{16} E_0^2 \]

\[ W_m = \frac{\mu}{4} \int_V (H_x H_x^* + H_z H_z^*) dV = \frac{\mu ab d}{16} E_0^2 \left( \frac{1}{Z_{TE}^2} + \frac{\pi^2}{k^2 \eta^2 a^2} \right) \]

Since \( Z_{TE} = k \eta / \beta \) and \( \beta = \beta_{10} = \sqrt{k^2 - (\pi/a)^2} \), we have

\[ \left( \frac{1}{Z_{TE}^2} + \frac{\pi^2}{k^2 \eta^2 a^2} \right) = \beta^2 + (\pi/a)^2 = \frac{1}{\eta^2} = \frac{\epsilon}{\mu} \]

and, therefore,

\[ W_e = W_m \]
RECTANGULAR WAVEGUIDE CAVITY RESONATORS

Unloaded Q of the TE$_{10\ell}$ mode

The power dissipated in the metallic boundary (assuming a good conductor) is obtained using the perturbation technique, i.e., using the field calculated in the lossless case and the formulas for the skin effect:

$$P_c = \frac{R_s}{2} \int_{\text{walls}} |H_t|^2 ds$$

where $R_s = \sqrt{\omega \mu_0 / 2\sigma}$ is the surface resistance.

$$P_c = \frac{R_s}{2} \left\{ 2 \int_{y=0}^{b} \int_{x=0}^{a} |H_x(z = 0)|^2 dx dy + 2 \int_{z=0}^{d} \int_{y=0}^{b} |H_z(x = 0)|^2 dy dz 
+ 2 \int_{z=0}^{d} \int_{x=0}^{a} \left[ |H_x(y = 0)|^2 + |H_z(y = 0)|^2 \right] dx dz \right\}$$

$$= \frac{R_s E_0^2 \lambda^2}{8\eta^2} \left( \ell^2 a b + \frac{bd}{a^2} + \frac{\ell^2 a}{2d} + \frac{d}{2a} \right),$$
RECTANGULAR WAVEGUIDE CAVITY RESONATORS

Unloaded Q of the TE_{10\ell} mode

The resulting unloaded Q-factor due to only conductor losses ($Q_c$) is

$$Q_c = \frac{2\omega_0 W_e}{P_c} = \frac{k^3 abd\eta}{4\pi^2 R_s} \frac{1}{[(\ell^2 ab/d^2) + (bd/a^2) + (\ell^2 a/2d) + (d/2a)]}$$

$$= \frac{(kad)^3 b\eta}{2\pi^2 R_s} \frac{1}{(2\ell^2 a^3 b + 2bd^3 + \ell^2 a^3 d + ad^3)}.$$
**RECTANGULAR WAVEGUIDE CAVITY RESONATORS**

Unloaded Q of the $\text{TE}_{10\ell}$ mode

The power dissipated in the dielectric (if any) with a dielectric constant $\epsilon = \epsilon_r \epsilon_0 (1 - j \tan \delta)$ is

$$P_d = \frac{1}{2} \int_V \bar{J} \cdot \bar{E}^* \, dv = \frac{\omega \epsilon''}{2} \int_V |\bar{E}|^2 \, dv = \frac{abd \omega \epsilon'' |E_0|^2}{8}$$

Therefore, the unloaded Q-factor due to only dielectric losses ($Q_d$) is

$$Q_d = \frac{2 \omega W_e}{P_d} = \frac{\epsilon'}{\epsilon''} = \frac{1}{\tan \delta}$$

It is interesting to notice that $Q_d$ does not depend on the shape of the cavity, but only on the intrinsic losses of the dielectric. Therefore this formula $Q_d$ is very general and applies to all the cavities fully filled with a dielectric, independently on their shape.
RECTANGULAR WAVEGUIDE CAVITY RESONATORS

Unloaded Q of the $\text{TE}_{10\ell}$ mode

Considering together the conductor and dielectric dissipated power (i.e., $P_{\text{diss}} = P_c + P_d$), it can easily be demonstrated that the overall unloaded Q-factor ($Q_0$) is a combination of the quality factors:

$$Q_0 = \left(\frac{1}{Q_c} + \frac{1}{Q_d}\right)^{-1}$$

In general, the dielectric losses are dominating, and $Q_d < Q_c$. This is the reason to avoid dielectrics in waveguide resonators.
CIRCULAR WAVEGUIDE CAVITY RESONATORS

A cavity based on a circular waveguide can be obtained by short circuiting the waveguide at both ends.

The dominant cylindrical cavity mode is the $\text{TE}_{111}$, derived from the $\text{TE}_{11}$ of the waveguide.

An example of application is the frequency meter in the photo: the cylindrical cavity is in between the two waveguides, and its resonance frequency is tuned until there is a transmission from the input to the output waveguide. Due to the high frequency selectivity (high-Q), this allows to determine the frequency with an high accuracy.
CIRCULAR WAVEGUIDE CAVITY RESONATORS

The field of the TE or TM propagating modes is

$$\vec{E}_t(\rho, \phi, z) = \vec{e}(\rho, \phi)(A^+ e^{-j\beta_{nm}z} + A^- e^{j\beta_{nm}z})$$

Imposing the perfect electric wall condition at $z = 0, d$ the resonant frequency and the resonant modal field can be obtained.

The resonant frequencies are

$$f_{nm\ell} = \frac{c}{2\pi \sqrt{\mu_r \epsilon_r}} \sqrt{\left( \frac{p'_{nm}}{a} \right)^2 + \left( \frac{\ell \pi}{d} \right)^2}$$  \hspace{1cm} \text{TE modes}

$$f_{nm\ell} = \frac{c}{2\pi \sqrt{\mu_r \epsilon_r}} \sqrt{\left( \frac{p_{nm}}{a} \right)^2 + \left( \frac{\ell \pi}{d} \right)^2}$$  \hspace{1cm} \text{TM modes}
Plotting the resonant frequencies as a function of the dimension, this design chart can be obtained.
Unloaded Q of the $\text{TE}_{nm\ell}$ mode

From it we obtain the stored energy

\[
W = 2W_e = \frac{\varepsilon}{2} \int_{z=0}^{d} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \left( |E_\rho|^2 + |E_\phi|^2 \right) \rho d\rho d\phi dz
\]

\[
= \frac{\varepsilon k^2 \eta^2 a^2 \pi d H_0^2}{4(p_{nm}')^2} \int_{\rho=0}^{a} \left[ J_n^2 \left( \frac{p_{nm}' \rho}{a} \right) + \left( \frac{na}{p_{nm}' \rho} \right)^2 J_n^2 \left( \frac{p_{nm}' \rho}{a} \right) \right] \rho d\rho
\]

\[
= \frac{\varepsilon k^2 \eta^2 a^4 H_0^2 \pi d}{8(p_{nm}')^2} \left[ 1 - \left( \frac{n}{p_{nm}'} \right)^2 \right] J_n^2 (p_{nm}').
\]
CIRCULAR WAVEGUIDE CAVITY RESONATORS

Unloaded $Q$ of the $TE_{nm\ell}$ mode

and the dissipated power

$$P_c = \frac{R_s}{2} \int_S |\tilde{H}_{\text{tan}}|^2 ds$$

$$= \frac{R_s}{2} \left\{ \int_{z=0}^{d} \int_{\phi=0}^{2\pi} \left[ |H_{\phi}(\rho = a)|^2 + |H_z(\rho = a)|^2 \right] a d\phi dz + 2 \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \left[ |H_{\rho}(z = 0)|^2 + |H_{\phi}(z = 0)|^2 \right] \rho d\rho d\phi \right\}$$

$$= \frac{R_s}{2} \pi H_0^2 J_n^2 (p'_{nm}) \left\{ \frac{da}{2} \left[ 1 + \left( \frac{\beta a n}{(p'_{nm})^2} \right)^2 \right] + \left( \frac{\beta a^2}{p'_{nm}} \right)^2 \left( 1 - \frac{n^2}{(p'_{nm})^2} \right) \right\}$$
The resulting unloaded Q-factor due to only conductor losses \((Q_c)\) is

\[
Q_c = \frac{\omega_0 W}{P_c} = \frac{(ka)^3 \eta ad}{4(p'_{nm})^2 R_s} \frac{1 - \left(\frac{n}{p'_{nm}}\right)^2}{\left\{ \frac{ad}{2} \left[ 1 + \left( \frac{\beta an}{(p'_{nm})^2} \right)^2 \right] + \left( \frac{\beta a^2}{p'_{nm}} \right)^2 \left( 1 - \frac{n^2}{(p'_{nm})^2} \right) \right\}}
\]
Unloaded Q of the $\text{TE}_{nml}$ mode

Normalized unloaded Q for various cylindrical cavity modes (air filled)
Unloaded Q of the TE$_{nm\ell}$ mode

The power dissipated in the dielectric $\varepsilon = \varepsilon_r \varepsilon_0 (1 - j \tan \delta)$ is

$$P_d = \frac{1}{2} \int_{V} \mathbf{J} \cdot \mathbf{E}^* \, dv = \frac{\omega \varepsilon''}{2} \int_{V} \left[ |E_\rho|^2 + |E_\phi|^2 \right] \, dv$$

$$= \frac{\omega \varepsilon'' k^2 \eta^2 \varepsilon_0 H_0^2 \pi d}{4 (p'_{nm})^2} \int_{\rho=0}^{a} \left[ \left( \frac{na}{p'_{nm \rho}} \right)^2 J_n^2 \left( \frac{p'_{nm \rho}}{a} \right) + J'_n^2 \left( \frac{p'_{nm \rho}}{a} \right) \right] \rho \, d\rho$$

$$= \frac{\omega \varepsilon'' k^2 \eta^2 \varepsilon_0 H_0^2}{8 (p'_{nm})^2} \left[ 1 - \left( \frac{n}{p'_{nm}} \right)^2 \right] J_n^2 (p'_{nm}).$$

Therefore, the unloaded Q-factor due to only dielectric losses ($Q_d$) is

$$Q_d = \frac{\omega W}{P_d} = \frac{\varepsilon}{\varepsilon''} = \frac{1}{\tan \delta}$$

As expected, $Q_d$ does not depend on the shape of the cavity, ad is the same as in the rectangular case.
Dielectric Resonators

A small disc or cube (or other shape) of (not metalized) dielectric material can be used as a microwave resonator.

Low loss and a high dielectric constant ($\varepsilon_r = 10 \div 100$) materials (e.g., BaO$_9$Ti$_4$, TiO$_2$), the field is confined within the dielectric (some field leakage leads to a small radiation loss lowering Q).

Smaller in size, cost, and weight than metallic cavity, easy to incorporate in microwave integrated circuits and to couple to planar transmission lines.

No conductor losses, but dielectric loss usually increases with dielectric constant. Q of up to several thousand can sometimes be achieved.

The resonant frequency can be mechanically tuned using an adjustable metal plate above the resonator.

Because of these desirable features, dielectric resonators have become key components for integrated microwave filters and oscillators.
Resonant frequencies of TE\(_{01\delta}\) mode

Due to the high-dielectric permittivity, the boundary condition on the surface of the dielectric can be approximated as a perfect magnetic wall (dual of the electric wall).

In fact, as a first order approximation of the reflection on the boundary is

\[
\Gamma = \frac{\sqrt{\varepsilon_r} - 1}{\sqrt{\varepsilon_r} + 1} \rightarrow 1 \quad \text{when} \quad \sqrt{\varepsilon_r} \rightarrow \infty
\]

Which is the opposite of the electric wall (\(\Gamma \rightarrow -1\))

Considering a cylindrical waveguide made of not metalized dielectric, the solution of the Helmoltz equation is dual with respect to the metalized counterpart (TE\(\rightarrow\)TM, TM\(\rightarrow\)TE, \(\vec{E} \rightarrow \vec{H}\), \(\vec{H} \rightarrow \vec{E}\)).
DIELECTRIC RESONATORS

Resonant frequencies of TE$_{01\delta}$ mode

The electric and magnetic field of the TE$_{01}$ mode of a cylindrical waveguide is

\[
E_\phi = \frac{j \omega \mu_0 H_0}{k_c} J_0'(k_c \rho) e^{\pm j \beta z},
\]

\[
H_\rho = \mp j \beta H_0 \frac{j}{k_c} J_0'(k_c \rho) e^{\pm j \beta z}.
\]

The propagation constant in the dielectric is

\[
\beta = \sqrt{\epsilon_r k_0^2 - k_c^2} = \sqrt{\epsilon_r k_0^2 - \left( \frac{p_{01}}{a} \right)^2}
\]

while in air the mode is attenuated with

\[
\alpha = \sqrt{k_c^2 - k_0^2} = \sqrt{\left( \frac{p_{01}}{a} \right)^2 - k_0^2}
\]

and the characteristic impedance is

\[
Z_d = \frac{E_\phi}{H_\rho} = \frac{\omega \mu_0}{\beta}
\]
## DIELECTRIC RESONATORS

### Resonant frequencies of \( \text{TE}_{01\delta} \) mode

| In the dielectric \((|z| < L/2)\) | In air \((|z| < L/2)\) |
|----------------------------------|---------------------|
| \( E_\phi = A J'_0(k_c \rho) \cos \beta z \) | \( E_\phi = B J'_0(k_c \rho) e^{-\alpha |z|} \) |
| \( H_\rho = \frac{-j A}{Z_d} J'_0(k_c \rho) \sin \beta z \) | \( H_\rho = \frac{\pm B}{Z_a} J'_0(k_c \rho) e^{-\alpha |z|} \) |

Matching the fields at the interfaces we have

\[
A \cos \frac{\beta L}{2} = B e^{-\alpha L/2},
\]

\[
\frac{-j A}{Z_d} \sin \frac{\beta L}{2} = \frac{B}{Z_a} e^{-\alpha L/2}
\]

which leads to

\[
-j Z_a \sin \frac{\beta L}{2} = Z_d \cos \frac{\beta L}{2}
\]

\[
\tan \frac{\beta L}{2} = \frac{\alpha}{\beta}
\]
**Dielectric Resonators**

Resonant frequencies of TE\(_{01\delta}\) mode

This solution is approximate (ignores fringing fields at the sides), with error on the order of 10% (not accurate enough for practical purposes). It illustrates the basic behavior of dielectric resonators, and more accurate solutions are available in the literature.

The unloaded Q of the resonator can be calculated by determining the stored energy (inside and outside the dielectric cylinder), and the power dissipated in the dielectric and possibly lost to radiation. If the radiation is small, the unloaded Q can be approximated as

\[ Q_0 \approx \frac{1}{\tan \delta} \]

as in the case of the metallic cavity resonators.
DIELECTRIC RESONATORS
**Excitation of Resonators**

Resonators are not useful unless they are coupled to external circuitry.

The way in which resonators can be coupled to transmission lines and waveguides depends on the type of resonator under consideration.

Some common coupling techniques (i.e., gap coupling and aperture coupling) will be discussed.

The coupling coefficient for a resonator connected to a feed line will be defined and discussed.

The determination of the $Q$ of a resonator from two-port measurement will be addressed.
Example of couplings:

- Microstrip resonator gap coupled to a microstrip feedline.
- Rectangular cavity resonator fed by a coaxial probe.
- Circular cavity resonator aperture coupled to a rectangular waveguide.
- Dielectric resonator coupled to a microstrip line.
EXCITATION OF RESONATORS

The level of coupling depends on the application.

If the high-Q of a waveguide cavity resonator is to be preserved (e.g., in a frequency meter), is usually loosely coupled to its feed guide.

On the contrary, resonators used in oscillators or in tuned amplifiers, may be tightly coupled in order to achieve maximum power transfer.

A measure of the level of coupling between a resonator and a feed is given by the coupling coefficient.

To obtain maximum power transfer between a resonator and a feed line, the resonator should be matched to the line at the resonant frequency; the resonator is then said to be critically coupled to the feed.
To define the *coupling coefficient*, let’s consider a series resonant circuit connected to a transmission line.

The input impedance near resonance of the series RLC is

\[
Z_{\text{in}} = R + j2L\Delta\omega = R + j\frac{2RQ_0\Delta\omega}{\omega_0}
\]

where

\[
Q_0 = \frac{\omega_0 L}{R}
\]
**COUPLING COEFFICIENT**

At resonance \((\omega = \omega_0, \Delta \omega = 0)\) the input impedance is \(Z_{in} = R\), therefore

\[
Q_0 = \frac{\omega_0 L}{Z_0}
\]

Moreover, assuming an infinite transmission line (or, equivalently, a matched generator), the external resistance seen from the resonator is \(R_{\text{external}} = Z_0\), and the external Q-factor become

\[
Q_e = \frac{\omega_0 L}{Z_0} = Q_0
\]

Therefore, the external and unloaded Q are identical at the critical coupling. In this case, the loaded Q is one half of the unloaded Q.
COUPLING COEFFICIENT

We can define the coupling coefficient

\[ g = \frac{Q_0}{Q_e} \]

which can be applied to both series \((g = Z_0/R)\) and parallel \((g = R/Z_0)\) resonant circuits connected to a transmission line of characteristic impedance \(Z_0\).

Three cases can be distinguished:

- \(g < 1\): resonator undercoupled to the feedline
- \(g = 1\): resonator critically coupled to the feedline
- \(g > 1\): resonator overcoupled to the feedline

The Smith chart sketch of the impedance loci for the series resonant circuit exemplifies the three cases.
Consider a $\lambda/2$ lossless open-circuited microstrip resonator proximity coupled to the open end of a microstrip transmission line.

The gap between the resonator and the microstrip line can be modeled as a series capacitor.

The normalized input impedance seen by the feedline is

$$z = \frac{Z}{Z_0} = -j \left( \frac{1}{\omega C} + \frac{Z_0 \cot \beta \ell}{Z_0} \right) = -j \left( \frac{\tan \beta \ell + b_c}{b_c \tan \beta \ell} \right)$$

where $b_c = Z_0 \omega C$ is the normalized susceptance of
Imposing the resonance condition $z = 0$, we have

$$\tan \beta \ell + b_c = 0$$

A graphical solution is shown in the plot.

Since usually $b_c \ll 1$, the first resonance $\omega_1$ is just below the resonance $\omega_0$ of the transmission line (corresponding to $\beta \ell = \pi$).

Therefore, the effect of the coupling is to lower the resonance frequency.
The goal is to represent the resonator around the first resonance $\omega_1$ as a RLC equivalent circuit.

Expanding $z$ in a Taylor series around $\omega_1$ we have

$$z(\omega) = z(\omega_1) + (\omega - \omega_1) \left. \frac{dz(\omega)}{d\omega} \right|_{\omega_1} + \cdots = (\omega - \omega_1) \left. \frac{dz(\omega)}{d(\beta \ell)} \right|_{\omega_1} \frac{d(\beta \ell)}{d\omega} + \cdots$$

where

$$\left. \frac{dz}{d(\beta \ell)} \right|_{\omega_1} = j \frac{\sec^2 \beta \ell}{\tan^2 \beta \ell} = j \frac{1 + \tan^2 \beta \ell}{\tan^2 \beta \ell} = j \frac{1 + b_c^2}{b_c^2} \approx j \frac{1}{b_c^2}$$

If the transmission line support a TEM mode, then

$$\frac{d(\beta \ell)}{d\omega} = \frac{d(\omega \ell/v_p)}{d\omega} = \frac{\ell}{v_p}$$

and, finally

$$z(\omega) \approx j \frac{\ell (\omega - \omega_1)}{b_c^2 v_p} \approx j \frac{\pi (\omega - \omega_1)}{\omega_1 b_c^2}$$
In conclusion, for a lossless transmission line the input impedance near the resonance is

\[ z(\omega) \approx \frac{j\ell(\omega - \omega_1)}{b^2_c v_p} \approx \frac{j\pi(\omega - \omega_1)}{\omega_1 b^2_c} \]

Since the impedance vanishes at the resonance, the resonator is equivalent to a RLC series.

To take into account losses for high-Q resonators, the perturbation technique can be used, replacing the resonant frequency \( \omega_1 \):

\[ \omega_1 \rightarrow \omega_1 \left(1 + \frac{j}{2Q_0}\right) \]

leading to

\[ z(\omega) = \frac{\pi}{2Q_0 b^2_c} + j \frac{\pi(\omega - \omega_1)}{\omega_1 b^2_c} \]

Remember that, for a transmission line, \( Q_0 = \frac{\beta}{2\alpha} \)
Since

\[ R = \frac{\pi Z_0}{2Q_0 b_c^2} \]

The coupling coefficient results:

\[ g = \frac{Z_0}{R} = \frac{2Q_0 b_c^2}{\pi} \]

Therefore

\[ b_c = \sqrt{\frac{\pi}{2Q_0}} \] gives the critical coupling \((R = Z_0)\)

\[ b_c < \sqrt{\frac{\pi}{2Q_0}} \] gives the undercoupling

\[ b_c > \sqrt{\frac{\pi}{2Q_0}} \] gives the overcoupling
Consider a $\lambda_g/2$ rectangular waveguide short circuited at the two ends, and coupled with another waveguide (maybe identical) through a small aperture at one end.

The small aperture acts as a shunt inductance, and the structure can be modeled by the following equivalent circuit.
Assuming $x_L = \frac{\omega L}{Z_0} \ll 1$, the normalized input admittance seen by the feedline is

$$y = Z_0 Y = -j \left( \frac{Z_0}{X_L} + \cot \beta \ell \right) = -j \left( \frac{\tan \beta \ell + x_L}{x_L \tan \beta \ell} \right)$$

Imposing the resonance condition $y = 0$, we have

$$\tan \beta \ell + x_L = 0$$

whose solutions provides the resonances

It is noted that the structure is **lossless**. Losses can be introduced later on by the perturbation technique.
The goal is again to represent the resonator around the first resonance $\omega_1$ as a resonant RLC circuit.

Expanding $y$ in a Taylor series around $\omega_1$ we have

$$y(\omega) = y(\omega_1) + (\omega - \omega_1) \frac{dy(\omega)}{d\omega} \bigg|_{\omega_1} + \cdots = (\omega - \omega_1) \frac{dy(\omega)}{d(\beta \ell)} \frac{d(\beta \ell)}{d\omega} \bigg|_{\omega_1} + \cdots$$

where

$$\frac{dy(\omega)}{d(\beta \ell)} = j \frac{\sec^2 \beta \ell}{\tan^2 \beta \ell} = j \frac{1 + \tan^2 \beta \ell}{\tan^2 \beta \ell} = j \frac{1 + x_L^2}{x_L^2} \approx j \frac{1}{x_L^2}$$

For the TE or TM mode of a waveguide it results

$$\frac{d\beta}{d\omega} = \frac{d}{d\omega} \sqrt{k_0^2 - k_c^2} = \frac{k_0}{\beta c}$$

and, finally

$$y(\omega) \approx \frac{j k_0 \ell}{x_L^2 \beta c} (\omega - \omega_1) \approx \frac{j \pi k_0}{x_L^2 \beta^2 c} (\omega - \omega_1)$$
In conclusion, for a lossless resonator the input admittance near the resonance is

\[ y(\omega) \simeq \frac{jk_0 \ell}{x_L^2 \beta c} (\omega - \omega_1) \simeq \frac{j \pi k_0}{x_L^2 \beta^2 c} (\omega - \omega_1) \]

Since the admittance vanishes at the resonance, the resonator is equivalent to a RLC parallel.

To take into account losses for high-Q resonators, the perturbation technique can be used, replacing the resonant frequency \( \omega_1 \):

\[ \omega_1 \rightarrow \omega_1 \left( 1 + \frac{j}{2Q_0} \right) \]

leading to

\[ y(\omega) \simeq \frac{\pi k_0 \omega_1}{2Q_0 \beta^2 c x_L^2} + j \frac{\pi k_0 (\omega - \omega_1)}{\beta^2 c x_L^2} \]

\[ \frac{1}{R} \]
The critical coupling (maximum power transfer) is achieved when \((R = Z_0)\), which corresponds to

\[
X_L = Z_0 \sqrt{\frac{\pi k_0 \omega_1}{2Q_0 \beta^2 c}}
\]

For a TE mode \(Z_0 = \frac{k_0 \eta_0}{\beta}\), and the coupling coefficient results:

\[
g = \frac{R}{Z_0} = \frac{\pi k_0 \omega_1}{2Q_0 \beta^2 c x_L^2} \frac{\beta}{k_0 \eta_0} = \frac{\pi \omega_1 \epsilon}{2Q_0 \beta x_L^2}
\]
Direct measurement of the unloaded $Q$ of a resonator is generally not possible because of the loading effect of the measurement system.

However, it is possible to determine unloaded $Q$ from measurements of the frequency response of the loaded resonator when it is connected to a transmission line.

Both one-port (reflection) and two-port (transmission) measurement techniques are possible.
Let us consider the (equivalent) RLC series circuit embedded in a two-port measurement.

The unloaded $Q$ is related to the loaded $Q$ and to the coupling factor:

\[
\frac{1}{Q_L} = \frac{1}{Q_e} + \frac{1}{Q_0} = \frac{1}{Q_0} \left( 1 + \frac{Q_0}{Q_e} \right) = \frac{1}{Q_0} (1 + g)
\]

\[
Q_0 = (1 + g) Q_L
\]

We need from measurements the loaded $Q$ and the coupling factor.
For the RLC series the unloaded Q is

\[ Q_0 = \frac{\omega_0 L}{R} \]

Moreover, the external Q is twice the one of the single line connection:

\[ Q_e = \frac{\omega_0 L}{2Z_0} \]

\[ g = \frac{Q_0}{Q_e} = \frac{2Z_0}{R} \]

Since for the RLC at the resonance we have \( Z = R \), it can be easily demonstrated that

\[ S_{21}(\omega_0) = \frac{2Z_0}{2Z_0 + Z(\omega_0)} = \frac{2Z_0}{2Z_0 + R} = \frac{g}{1 + g} \]

\[ g = \frac{S_{21}(\omega_0)}{1 - S_{21}(\omega_0)} \]

Where \( S_{21} \) is the measured transmission parameter, which is purely real at the resonance, and provides the coupling factor.
The determination of the loaded $Q$ is done from the transmission measurement around the resonance:

$$Q_L = \frac{f_0}{\text{BW}}$$
The same reasoning can be repeated for an RLC parallel resonator. The only difference is that

$$g = \frac{1 - S_{21}(\omega_0)}{S_{21}(\omega_0)}$$

Nothing changes for the determination of the loaded Q.
CAVITY PERTURBATIONS

Cavity resonators are often modified by making small changes in their shape, or by introducing small pieces of dielectric or metallic materials.

For example, the resonant frequency of a cavity resonator can be easily tuned with a small screw (dielectric or metallic) that enters the cavity volume, or by changing the size of the cavity with a movable wall.

Another application involves the determination of dielectric constant of materials by measuring the shift in resonant frequency when a small dielectric sample is introduced into the cavity.

In some cases, the effect of such perturbations on the cavity performance can be calculated exactly, but often approximations must be made. One useful technique for doing this is the perturbational method, which assumes that the actual fields of a cavity with a small shape or material perturbation are not greatly different from those of the unperturbed cavity.
CAVITY PERTURBATIONS

In the case of material perturbation, starting from the Maxwell’s equations it can be demonstrated that the shift of the resonant frequency is given by

\[
\frac{\omega - \omega_0}{\omega} = \frac{-\int_{V_0} (\Delta \varepsilon \vec{E} \cdot \vec{E}_0^* + \Delta \mu \vec{H} \cdot \vec{H}_0^*) \, dv}{\int_{V_0} (\varepsilon \vec{E} \cdot \vec{E}_0^* + \mu \vec{H} \cdot \vec{H}_0^*) \, dv}
\]

(exact expression)

If we assume \( \vec{E} \approx \vec{E}_0 \) and \( \vec{H} \approx \vec{H}_0 \), and approximate \( \omega \) with \( \omega_0 \) in the denominator, we obtain

\[
\frac{\omega - \omega_0}{\omega_0} \approx \frac{-\int_{V_0} (\Delta \varepsilon |\vec{E}_0|^2 + \Delta \mu |\vec{H}_0|^2) \, dv}{\int_{V_0} (\varepsilon |\vec{E}_0|^2 + \mu |\vec{H}_0|^2) \, dv}
\]

(approximate expression)
CAVITY PERTURBATIONS

In the case of shape perturbation, starting from the Maxwell’s equations it can be demonstrated that the shift of the resonant frequency is given by

$$\omega - \omega_0 = \frac{-j \int_{\Delta S} \mathbf{E}_0^* \times \mathbf{H} \cdot d\mathbf{s}}{\int_V (\varepsilon \mathbf{E} \cdot \mathbf{E}_0^* + \mu \mathbf{H} \cdot \mathbf{H}_0^*) dv}$$

(exact expression)

If we assume $\mathbf{E} \approx \mathbf{E}_0$ and $\mathbf{H} \approx \mathbf{H}_0$, we obtain

$$\frac{\omega - \omega_0}{\omega_0} \approx \frac{\int_{\Delta V} (\mu |\mathbf{H}_0|^2 - \varepsilon |\mathbf{E}_0|^2) dv}{\int_{V_0} (\mu |\mathbf{H}_0|^2 + \varepsilon |\mathbf{E}_0|^2) dv}$$

(approximate expression)

The frequency shift is related to the change in the stored energy.