propagazione in sistemi dispersivi

$$
\begin{aligned}
f_{\mathrm{o}}(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_{\mathrm{o}}(\omega)\, e^{j\omega t}\, d\omega \\
F_{\mathrm{o}}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_{\mathrm{o}}(t)\, e^{-j\omega t}\, dt
\end{aligned}
$$

$$
f_{\mathrm{o}}(t) = \sqrt{\frac{2}{\pi}}\, \mathrm{Re}\left(\int_{0}^{\infty} F_{\mathrm{o}}(\omega)\, e^{j\omega t}\, d\omega\right)
$$

$$f(0,t) = f_o(t)$$

$$f(z,t) = \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left(\int_0^\infty F(z,\omega)\, e^{j\omega t}\, d\omega\right)$$

$$F(z,\omega) = F(0,\omega)\, e^{-\gamma(\omega)z} = F_o(\omega)\, e^{-\gamma(\omega)z}$$

$$\boxed{f(z,t) = \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left(\int_0^\infty F_o(\omega)\, e^{j\omega t-\gamma z}\, d\omega\right)}$$

linea non dispersiva

$$\gamma(\omega) = j\omega / v_f$$

$$\alpha = 0$$

$$\beta = \omega / v_f$$

$$\begin{aligned} f(z,t) &= \sqrt{\frac{2}{\pi}}\, \mathrm{Re}\left(\int_0^\infty F_o(\omega)\, e^{j\omega(t - z/v_f)}\, d\omega \right) \\ &= f_o(t - z/v_f) \end{aligned}$$

segnale a banda limitata in una linea dispersiva

$$
\begin{aligned}
f_o(t) &= \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left(\int_{\omega_o-\Delta\omega}^{\omega_o+\Delta\omega} F_o(\omega)\, e^{j\omega t}\, d\omega\right) \\
&= \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left(e^{j\omega_o t}\int_{-\Delta\omega}^{\Delta\omega} F_o(\omega_o+\xi)\, e^{j\xi t}\, d\xi\right)
\end{aligned}
$$

$$
g(t)\, e^{j\chi(t)} = \sqrt{\frac{2}{\pi}}\int_{-\Delta\omega}^{\Delta\omega} F_o(\omega_o+\xi)\, e^{j\xi t}\, d\xi
$$

$$
f_o(t) = g(t)\cos\left(\omega_o t + \chi(t)\right)
$$

$$\gamma(\omega) \;=\; \alpha_o + j\,\frac{\omega_o}{v_f} + j\,\frac{1}{v_g}\,(\omega - \omega_o) + \gamma_r(\omega)$$

$\alpha_o = \mathrm{Re}(\gamma(\omega_o))$	è la *costante di attenuazione* della linea alla pulsazione ω_o
$\dfrac{\omega_o}{v_f} = \mathrm{Im}(\gamma(\omega_o))$	è la costante di fase alla pulsazione ω_o e v_f è la *velocità di fase* alla stessa pulsazione
$\dfrac{1}{v_g} = \mathrm{Im}\left(\left.\dfrac{\partial\,\gamma(\omega)}{\partial\omega}\right\rvert_{\omega_o}\right)$	v_g è, per definizione, la *velocità di gruppo* alla pulsazione ω_o
$\gamma_r(\omega)$	è quello che resta a $\gamma(\omega)$ dopo l'estrazione dei termini messi in evidenza

$$j\omega t - \gamma z \;=\; -\alpha_o z + j\omega_o(t - z/v_f) + j(\omega - \omega_o)(t - z/v_g) - \gamma_r(\omega)\,z$$

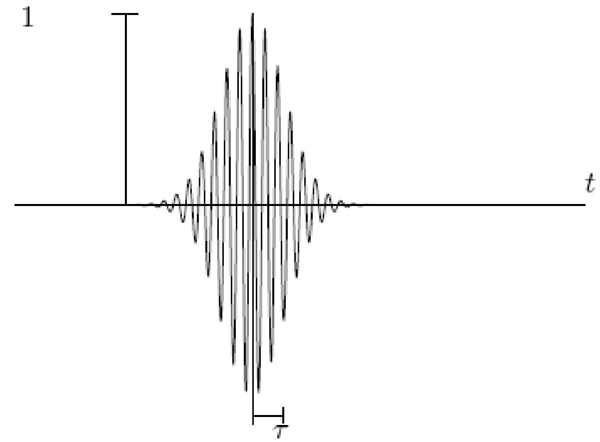
$$
\begin{aligned}
f(z,t) &= \sqrt{\frac{2}{\pi}}\, e^{-\alpha_o z}\, \mathrm{Re}\left(e^{j\omega_o(t-z/v_f)} \int_{\omega_o-\Delta\omega}^{\omega_o+\Delta\omega} \left(F_o(\omega)\, e^{-\gamma_r(\omega) z} \right)\, e^{j(\omega-\omega_o)(t-z/v_g)}\, d\omega \right) \\
&= \sqrt{\frac{2}{\pi}}\, e^{-\alpha_o z}\, \mathrm{Re}\left(e^{j\omega_o(t-z/v_f)} \int_{-\Delta\omega}^{\Delta\omega} \left(F_o(\omega_o+\xi)\, e^{-\gamma_r(\omega_o+\xi) z} \right)\, e^{j\xi(t-z/v_g)}\, d\xi \right)
\end{aligned}
$$

nelle sezioni in cui vale l’approssimazione

$$
F_o(\omega_o+\xi)\, e^{-\gamma_r(\omega_o+\xi) z} \approx F_o(\omega_o+\xi)
$$

$$
f(z,t) \approx e^{-\alpha_o z}\, g(t-z/v_g)\, \cos\left(\omega_o(t-z/v_f) + \chi(t-z/v_g)\right)
$$

impulso gaussiano

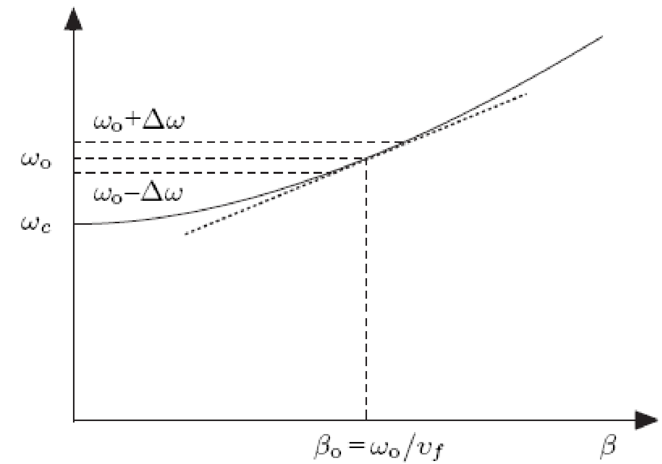


$$f_o(t) = e^{-\frac{1}{2}\,t^2\tau^{-2}} \;\cos\omega_o t$$

$$\begin{aligned}
F_o(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_o(t) e^{-j\omega t}\,dt \\
&= \frac{1}{2\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,t^2\tau^{-2}} \left(e^{j(\omega_o-\omega)t} + e^{-j(\omega_o+\omega)t}\right)\,dt \\
&= \frac{1}{2\sqrt{2\pi}} \left(G(\tau^{-1}, \omega_o - \omega) + G(\tau^{-1}, -\omega_o - \omega)\right) \\
&= \underbrace{\frac{\tau}{2}\,e^{-\frac{1}{2}\,\tau^2(\omega-\omega_o)^2}}_{F_1(\omega)} + \underbrace{\frac{\tau}{2}\,e^{-\frac{1}{2}\,\tau^2(\omega+\omega_o)^2}}_{F_1(-\omega)}
\end{aligned}$$

$$G(a,b) = \int_{-\infty}^{\infty} e^{-\frac{1}{2}\,a^2x^2}\; e^{j\,b\,x} dx = \sqrt{\frac{2\pi}{a^2}}\; e^{-\frac{1}{2}\,b^2a^{-2}} \qquad \mathrm{Re}(a^2) > 0$$

$$f(z,t) = \sqrt{\frac{2}{\pi}}\,\mathrm{Re}\left(\int_{-\infty}^{\infty} F_1(\omega)\, e^{j\omega t}\, d\omega\right)$$



$$\gamma_r(\omega) = \frac{j}{2}\,\beta_2(\omega-\omega_o)^2$$

$$\beta_2 = \mathrm{Im}\left(\left.\frac{\partial^2\,\gamma(\omega)}{\partial\omega^2}\right|_{\omega_o}\right)$$

$$f(z,t) = e^{-\alpha_o z}\,\frac{e^{-\frac{(t-z/v_g)^2}{2\,\tau^2(1+\delta^2 z^2)}}}{\sqrt[4]{1+\delta^2 z^2}}\,\cos\left(\omega_o(t-z/v_f) + \frac{(t-z/v_g)^2\delta z}{2\,\tau^2(1+\delta^2 z^2)} - \frac{1}{2}\arctan(\delta z)\right)$$

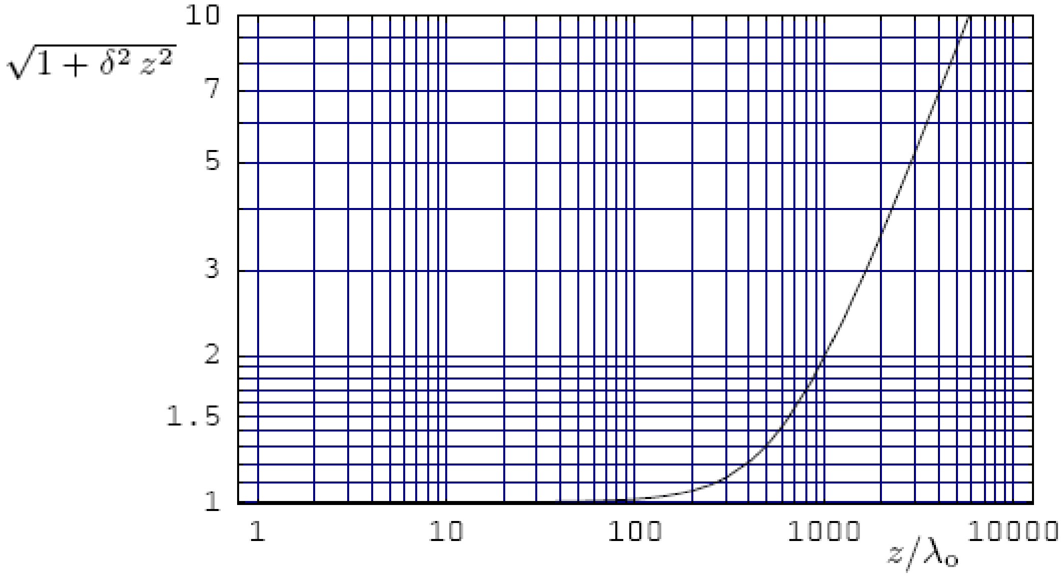
$$\delta = \beta_2\,/\,\tau^2$$

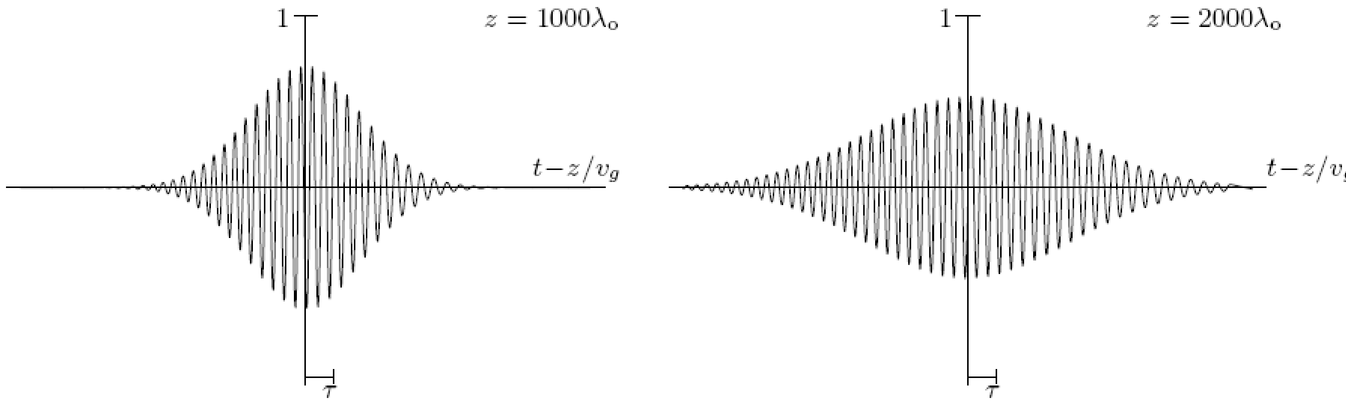
$$\beta = \frac{\omega}{c}\sqrt{1 - \frac{\omega_c^2}{\omega^2}}$$

$$\omega_o = 1.5\,\omega_c$$

$$\tau = 10\,T$$

$$\begin{aligned}
v_f &= c\sqrt{1 - \frac{\omega_c^2}{\omega_o^2}} && \approx\; 1.34\,c \\
v_g &= \frac{c^2}{v_f} && \approx\; 0.75\,c \\
\beta_2 &= -\frac{\omega_c^2}{2\,c\,\omega_o^3}\left(1 - \frac{\omega_c^2}{\omega_o^2}\right)^{-3/2} && \approx -\,0.17\,T^2/\lambda_o \\
\delta &= \frac{\beta_2}{\tau^2} && \approx -\,1.7\,10^{-3}/\lambda_o
\end{aligned}$$


$\sqrt{1+\delta^2 z^2}$
10
7
5
3
2
1.5
1
1
10
100
1000
10000
z/λ_o


1
$z = 1000\lambda_o$
$t-z/v_g$
τ
1
$z = 2000\lambda_o$
$t-z/v_g$
τ