Lecture 10

THE FINITE ELEMENT METHOD (FEM)

INTRODUCTION

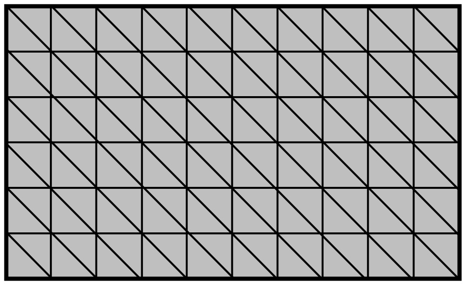
The finite-element method (FEM) is a powerful and versatile technique, adopted for the solution of several electromagnetic problems.

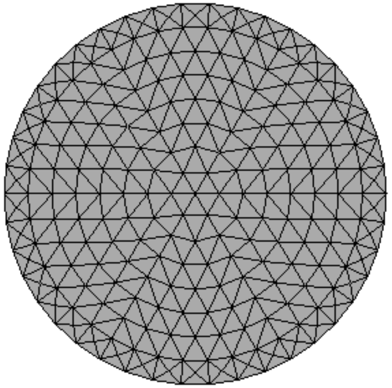
> The FEM is a numerical technique for obtaining approximate solutions to boundary-value problems. It is typically formulated in the frequency domain, and it can be applied to the modeling of complex regions, filled with an inhomogeneous medium.

The FEM originated in the 1940's from the need for solving complex elasticity and structural analysis problems in civil and aeronautical engineering. It was applied to the electromagnetic modeling by P. Silvester in 1969 for the first time.

INTRODUCTION

The investigation domain is divided in a number of subdomains, which span the entire space with no overlap. Thanks to this feature, the FEM is suitable to the modeling of arbitrary domains with inhomogeneous medium.





In each subdomain (or element) the unknown function is approximated by interpolating functions, which depend on the value of the unknown in the element. Continuity conditions are imposed between adjacent elements.

INTRODUCTION

The FEM method is based on the transformation of a differential problem with boundary conditions into an algebraic matrix problem.

The implementation of the FEM involves four steps:

1. domain segmentation in non-overlapping sub-domains (finite elements)
2. local definition of interpolating functions
3. formulation of the problem (by Rayleigh-Ritz or Galerkin method)
4. solution of the matrix problem

Reference books:

1. M.N.O. Sadiku, *Numerical techniques in Electromagnetics*, CRC Press, 2000.
2. Jianming Jin, *The finite element method in electromagnetics*, J. Wiley & Sons, 1993.

Variational methods include a wide class of techniques for the solution of partial differential equations in mathematical physics and engineering. They permit to replace the solution of a differential equation with the minimization of some integrals.

Variational methods give accurate results with limited demands on memory allocation and computing time.

They can be classified into two groups:

- direct methods (e.g. the classical Rayleigh-Ritz method);
- indirect methods (e.g. Galerkin and least square methods).

Preliminarily, we briefly discuss the Rayleigh-Ritz and Galerkin methods.

RAYLEIGH-RITZ METHOD / 1

Let us consider the boundary value problem

$$L\phi = f \quad \text{in} \quad \Omega$$

where L is a differential operator, ϕ is an unknown function that satisfies the boundary condition, and f is the excitation function (known).

If the operator L is self-adjoint, that is

$$\langle L\phi, \psi \rangle = \langle \phi, L\psi \rangle$$

and positive definite, that is

$$\langle L\phi, \phi \rangle \begin{cases} > 0 & \phi \neq 0 \\ = 0 & \phi = 0 \end{cases}$$

where the inner product is defined as $\langle \phi, \psi \rangle = \int_\Omega \phi \, \psi^* d\Omega$, then the solution of the boundary value problem can be obtained by minimizing the functional

$$F(\phi) = \frac{1}{2}\langle L\phi, \phi \rangle - \frac{1}{2}\langle \phi, f \rangle - \frac{1}{2}\langle f, \phi \rangle$$

RAYLEIGH-RITZ METHOD / 2

The unknown function ϕ can be approximated by the expansion:

$$\phi \cong \sum_{j=1}^{N} c_j v_j = [c]^T [v] = [v]^T [c]$$

where v_j basis functions defined in Ω e c_j are unknown coefficients. After replacing in the expression of the functional, it results:

$$F = \frac{1}{2}[c]^T \left(\int_\Omega [v] L [v]^T d\Omega \right) [c] - [c]^T \int_\Omega [v] f \, d\Omega$$

To minimize the functional F, we let its partial derivative with respect to c_j vanish. This leads to a set of linear algebraic equations

$$\frac{\partial F}{\partial c_i} = \frac{1}{2}\left(\int_\Omega v_i L [v]^T d\Omega \right)[c] + \frac{1}{2}[c]^T \left(\int_\Omega [v] L v_i d\Omega \right) - \int_\Omega v_i f \, d\Omega$$

$$= \frac{1}{2}\sum_{j=1}^{N} c_j \left\{ \int_\Omega \left(v_i L v_j + v_j L v_i \right) d\Omega \right\} - \int_\Omega v_i f \, d\Omega = 0 \qquad i = 1, 2, 3, \ldots, N$$

RAYLEIGH-RITZ METHOD / 3

The system of N equation can be recast in matrix form

$$[S][c]=[b]$$

where

$$S_{ij}=\frac{1}{2}\int_{\Omega}\left(v_i L v_j + v_j L v_i\right)d\Omega \qquad b_i=\int_{\Omega} v_i f\, d\Omega$$

Since the operator L is self-adjoint, it results:

$$S_{ij}=\int_{\Omega} v_i L v_j \, d\Omega$$

The solution of the matrix equation yields the unknown coefficients c_j and, consequently, the unknown function f.

GALERKIN METHOD / 1

Let us consider the same boundary value problem

$$L\phi = f \quad \text{in } \Omega$$

where L is a linear differential operator (no need it is self-adjoint).

The unknown function ϕ can be express as a linear combination of basis functions v_j:

$$\phi \cong \sum_{j=1}^{N} c_j v_j = [c]^T [v] = [v]^T [c]$$

and we use test functions $w_i = v_i$.

After replacing the approximated expression of ϕ in the equation, both member are multiplied by the test functions and integrated on the domain Ω, thus providing a system of N equations:

$$\int_{\Omega} v_i \, L [v]^T [c] \, d\Omega = \int_{\Omega} v_i \, f \, d\Omega \qquad i = 1, 2, 3, \ldots, N$$

The system of N equations can be recast in matrix form:

$$[S][c] = [b]$$

where

$$S_{ij} = \int_{\Omega} v_i \, L \, v_j \, d\Omega \qquad b_i = \int_{\Omega} v_i \, f \, d\Omega$$

The resulting problem is formally identical to the one obtained by applying the Rayleigh-Ritz method, but in this case no hypothesis that the operator is self-adjoint is needed.

It is finally observed that matrix S is symmetric if the operator L is self-adjoint.

EXAMPLE / 1

Let us consider the simple example of a functional equation:

$$\frac{d^2\phi}{dx^2} = x + 1 \qquad 0 \le x \le 1$$
$$\phi(0) = 0$$
$$\phi(1) = 1$$

The functional associated to this equation is the following:

$$F(\phi) = \frac{1}{2}\int_0^1 \left(\frac{d\phi}{dx}\right)^2 dx + \int_0^1 (x+1)\,\phi\,dx$$

The exact solution of this equation is:

$$\phi(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x$$

Solution by the Rayleigh-Ritz method

The unknown can be represented by the polynomial function:

$$\phi(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3$$

By applying the boundary conditions:

$$\phi(0) = c_1 = 0 \qquad c_1 = 0$$

$$\phi(1) = c_2 + c_3 + c_4 = 1 \qquad c_2 = 1 - c_3 - c_4$$

After incorporating the boundary conditions, the polynomial function can be expressed as:

$$\phi(x) = x + c_3\left(x^2 - x\right) + c_4\left(x^3 - x\right)$$

EXAMPLE / 3

After substituting in the expression of the functional and calculating the integrals, it results:

$$F(\phi) = \frac{2}{5}c_4^2 + \frac{1}{6}c_3^2 + \frac{1}{2}c_3 c_4 - \frac{23}{60}c_4 - \frac{1}{4}c_3 + \frac{4}{3}$$

The functional is minimized with respect to c_3 and c_4. By setting to zero the partial derivatives, we can obtain a system of equations:

$$\begin{cases} \dfrac{\partial F}{\partial c_3} = \dfrac{1}{3}c_3 + \dfrac{1}{2}c_4 - \dfrac{1}{4} = 0 \\ \dfrac{\partial F}{\partial c_4} = \dfrac{1}{2}c_3 + \dfrac{4}{5}c_4 - \dfrac{23}{60} = 0 \end{cases}$$

The solution of the system provides c_3=1/2 and c_4=1/6 (exact solution).

Solution by the Galerkin method

We approximate the unknown function as in the previous case:

$$\phi(x) = x + c_3\left(x^2 - x\right) + c_4\left(x^3 - x\right)$$

where c_3 and c_4 are unknown coefficients to determine.

We use the following test functions:

$$w_1 = x^2 - x \qquad\qquad w_2 = x^3 - x$$

By performing the test, we derive the following two equations:

$$\int_0^1 w_i \frac{d^2}{dx^2}(\phi)\,dx = \int_0^1 w_i\left(x+1\right)dx \qquad\qquad i = 1, 2$$

EXAMPLE / 5

University of Pavia

By replacing the expressions of basis and test functions:

$$\begin{cases} \int_0^1 \left(x^2 - x\right) \frac{d^2}{dx^2}\left[x + c_3\left(x^2 - x\right) + c_4\left(x^3 - x\right)\right] dx = \int_0^1 \left(x^2 - x\right)\left(x + 1\right) dx \\ \int_0^1 \left(x^3 - x\right) \frac{d^2}{dx^2}\left[x + c_3\left(x^2 - x\right) + c_4\left(x^3 - x\right)\right] dx = \int_0^1 \left(x^3 - x\right)\left(x + 1\right) dx \end{cases}$$

After computing the integrals, we obtain:

$$\begin{cases} \frac{1}{3} c_3 + \frac{1}{2} c_4 = \frac{1}{4} \\ \frac{1}{2} c_3 + \frac{4}{5} c_4 = \frac{23}{60} \end{cases}$$

The solution of the system of equations yields c_3=1/2 and c_4=1/6 (exact solution).

EXAMPLE / 6

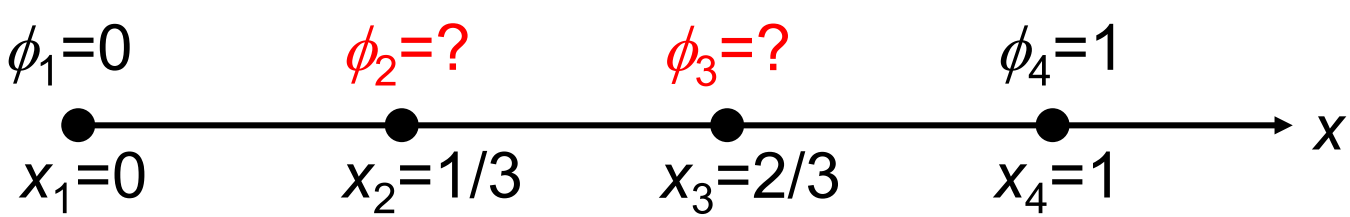
Solution by the FEM method

The interval $0<x<1$ is subdivided into three sub-intervals, and subdomain basis functions are defined in each sub-interval.

$$\phi(x) = \phi_i \frac{x_{i+1} - x}{x_{i+1} - x_i} + \phi_{i+1} \frac{x - x_i}{x_{i+1} - x_i} \qquad (x_i \le x \le x_{i+1}, i = 1..3)$$

These basis functions are linear and moreover $\phi_i = \phi(x_i)$.

As ϕ_1 and ϕ_4 are known (boundary conditions), the unknowns are ϕ_2 and ϕ_3.



By replacing the expressions of the basis functions in the functional and integrating, we obtain:

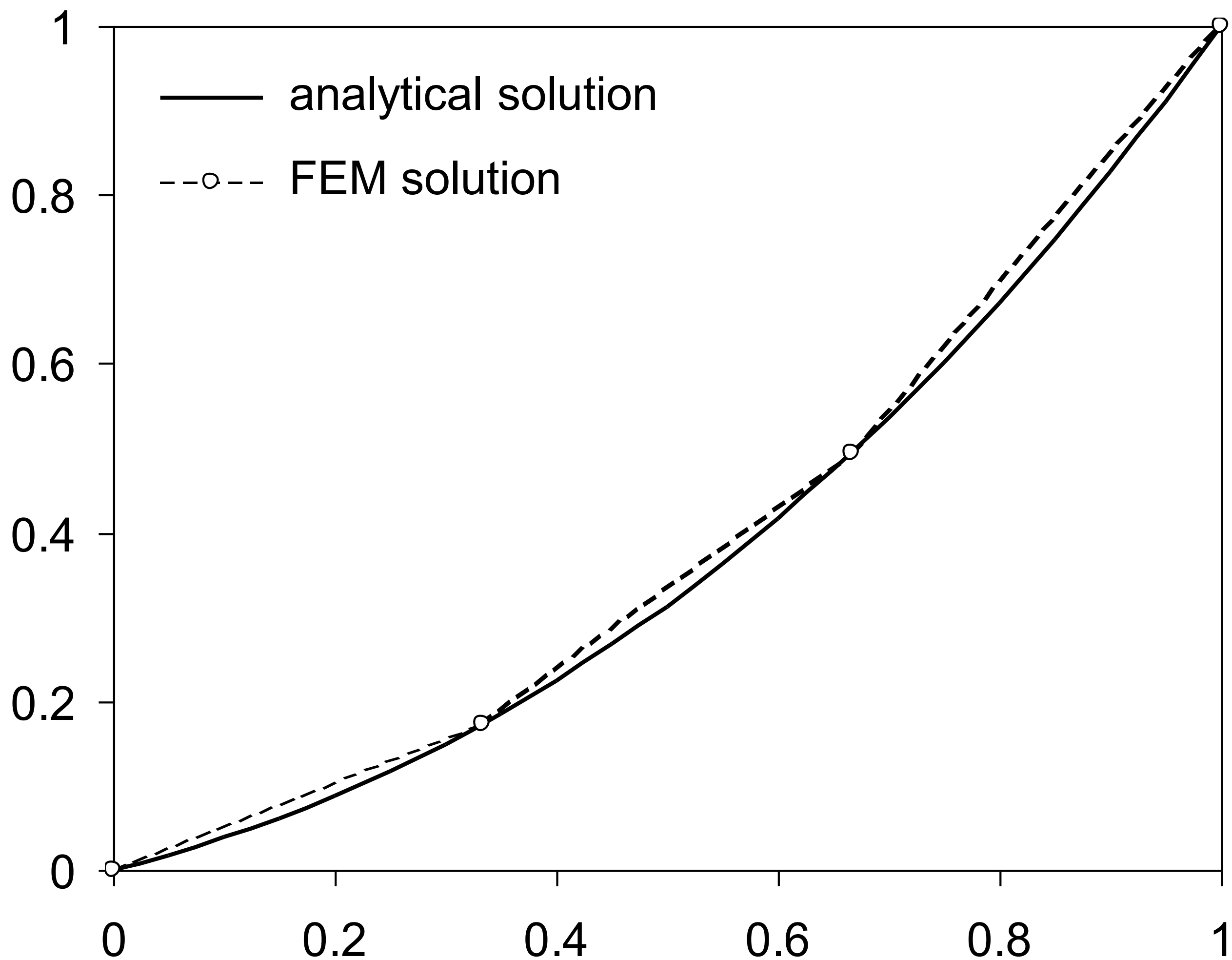
$$F(\phi) = 3\phi_2^2 + 3\phi_3^2 - 3\phi_2\phi_3 + \frac{4}{9}\phi_2 - \frac{22}{9}\phi_3 + \frac{49}{27}$$

The partial derivatives with respect to ϕ_2 and ϕ_3 are set to zero, and they provide the system of equations to solve:

$$\begin{cases} \dfrac{\partial F}{\partial \phi_2} = 6\phi_2 - 3\phi_3 + \dfrac{4}{9} = 0 \\ \dfrac{\partial F}{\partial \phi_3} = 6\phi_3 - 3\phi_2 - \dfrac{22}{9} = 0 \end{cases}$$

The solution of the system of equations provides ϕ_2=14/81 and ϕ_3=40/81 (approximate solution!).

EXAMPLE / 8



NOTE: In this case the solution is exact in points $x=x_2$ and $x=x_3$, but this is not always verified!