Lecture 11

THE FINITE ELEMENT METHOD (FEM)

INTRODUCTION

In this lecture, the basic steps needed for the implementation of the FEM are discussed.

The implementation of the FEM involves four steps:

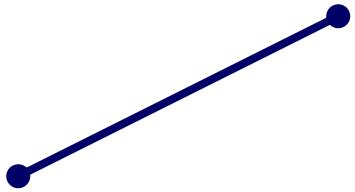
1. domain segmentation in non-overlapping sub-domains (finite elements)
2. local definition of interpolating functions
3. formulation of the problem (by Rayleigh-Ritz or Galerkin method)
4. solution of the matrix problem

Definition of the Mesh

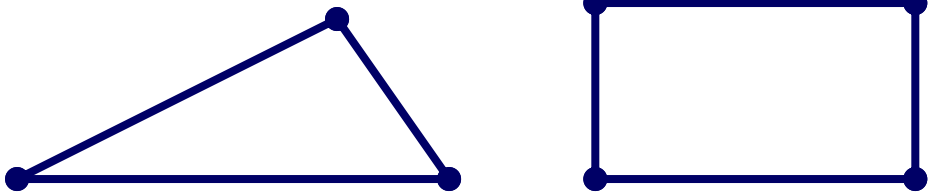
MESHING TECHNIQUES / 1

The segmentation of the domain (**mesh**) is the first operation to be performed in the FEM analysis. Specialized software tools perform the mesh in 2D and 3D case, optimizing the geometry representation and the numbering of the nodes (in order to obtain **band matrices**).

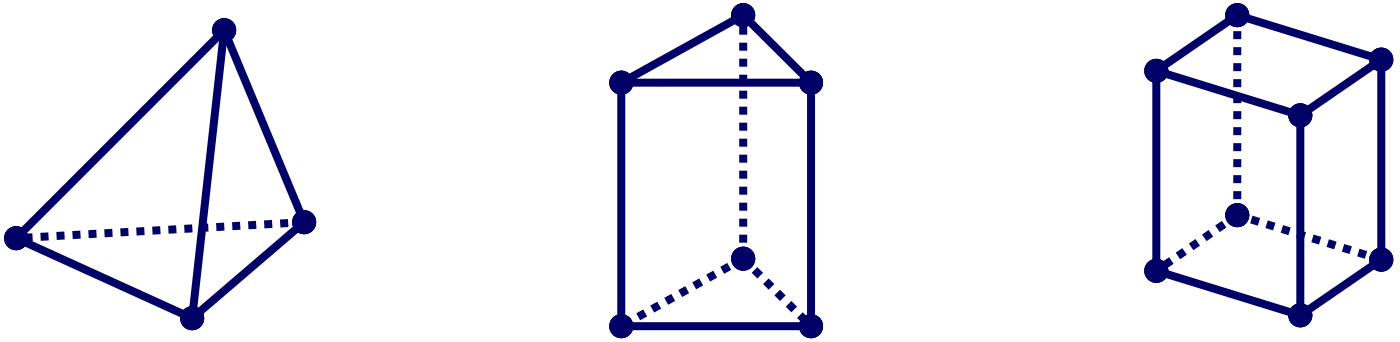
1D



2D



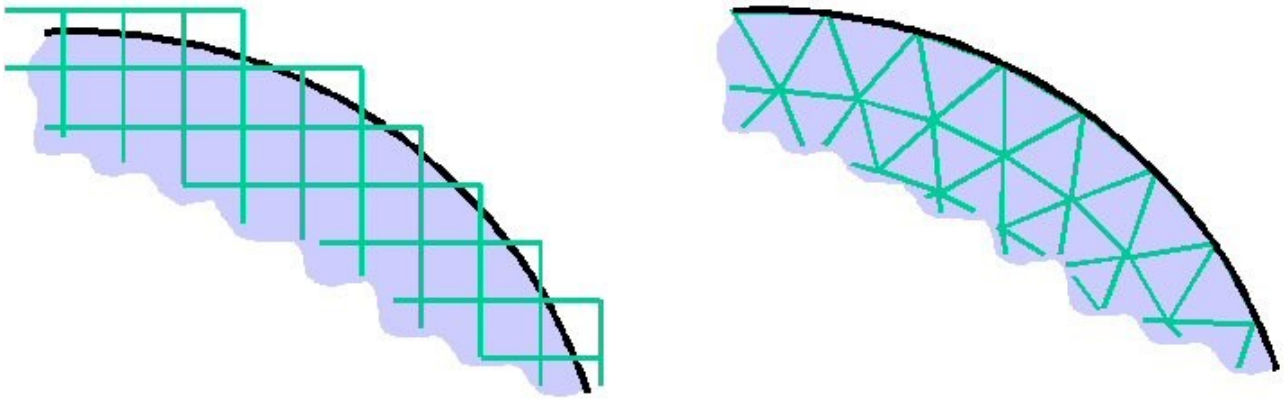
3D



MESHING TECHNIQUES / 2

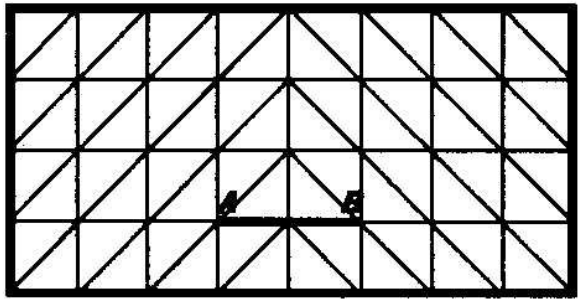
Finite elements can present different shapes, but in all cases their size is much smaller than wavelength (typically **λ/5 to λ/20**).

The accuracy of the solution, as well as computing time and memory allocation increase with the number of elements.

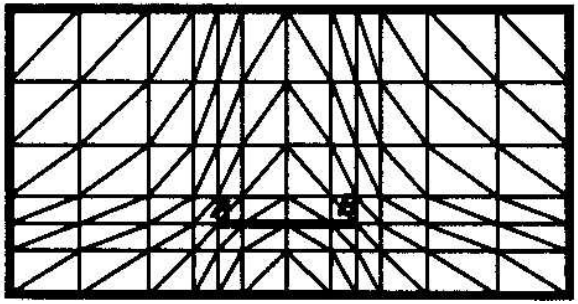


Triangular (in 2D cases) or tetrahedral (in 3D cases) elements are usually preferred, because they provide a better representation of the geometry in the case of curved or oblique boundaries.

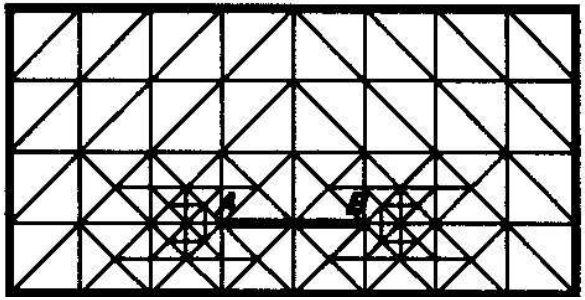
MESHING TECHNIQUES / 3



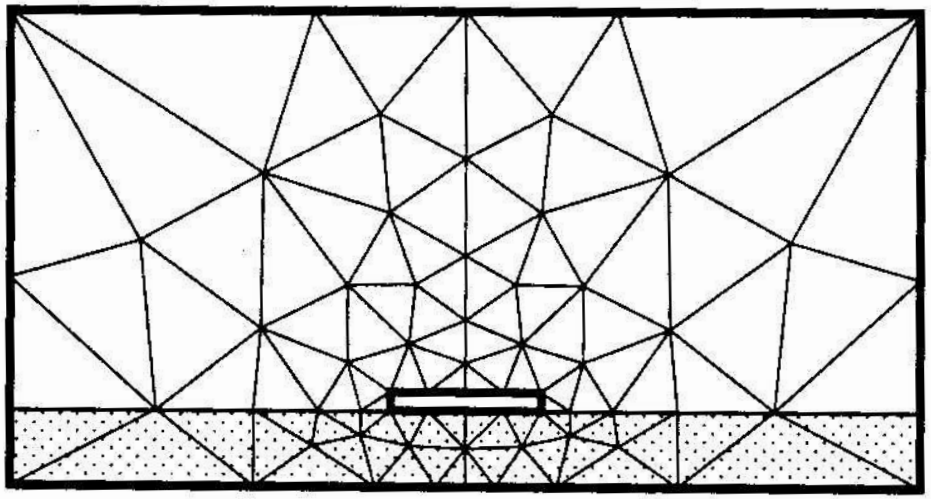
uniform



gradual



adaptive



unstructured

MESHING TECHNIQUES / 4

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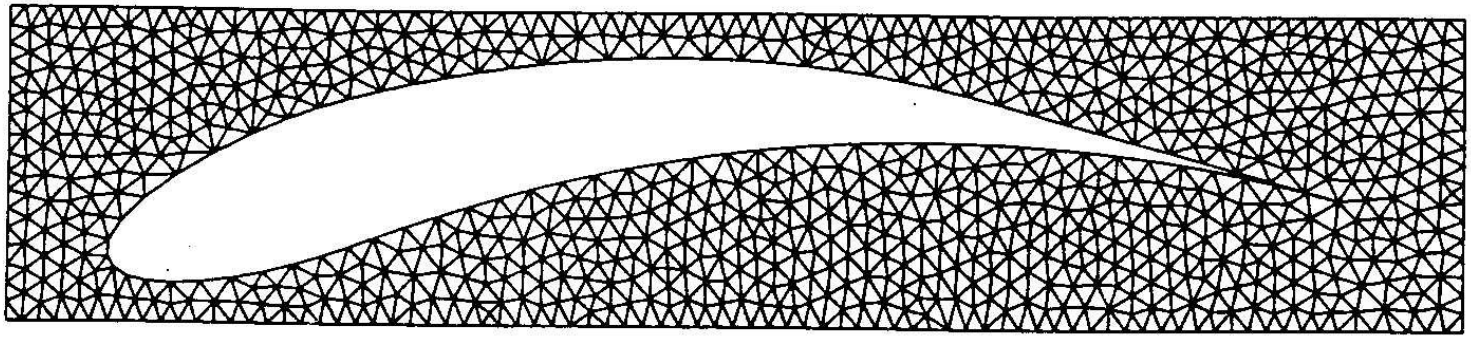
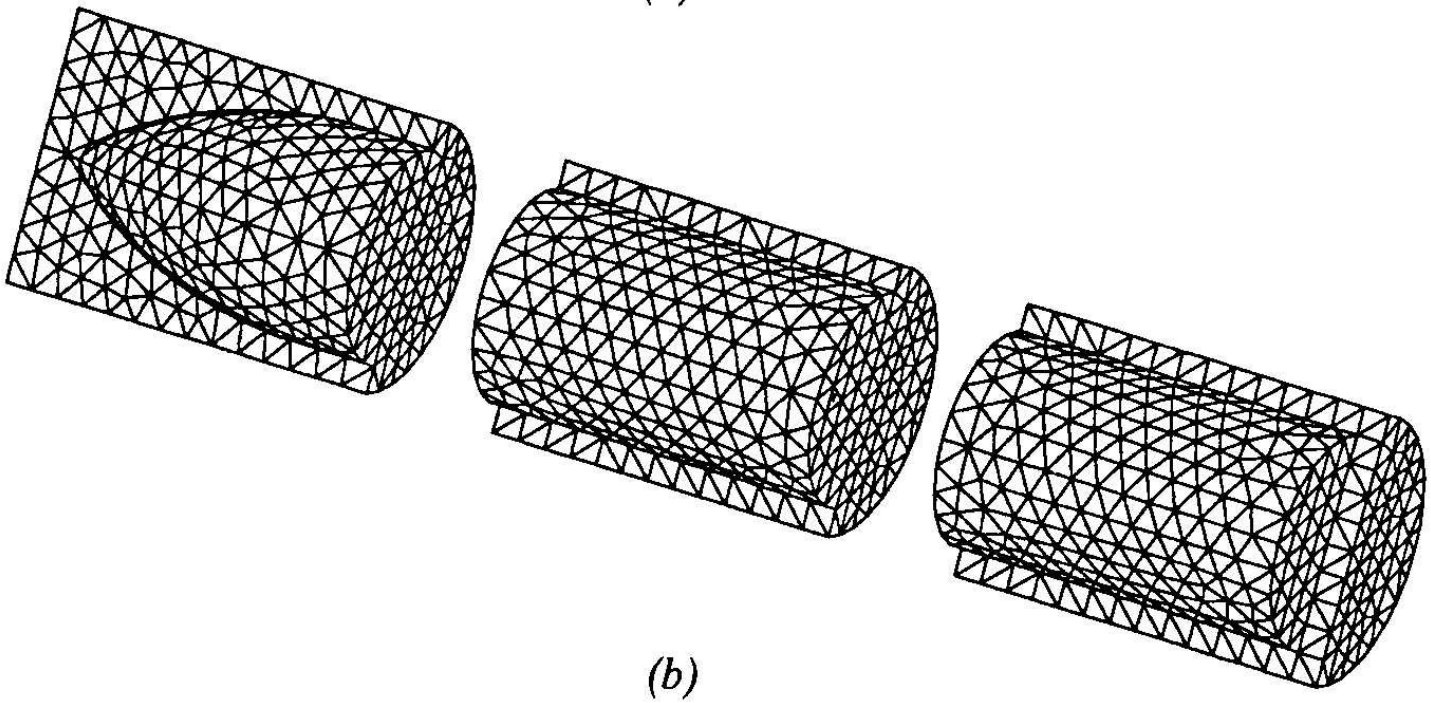
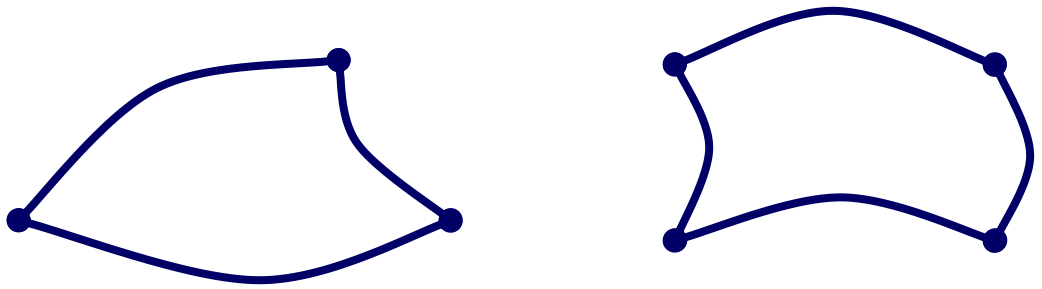


(a)



(b)

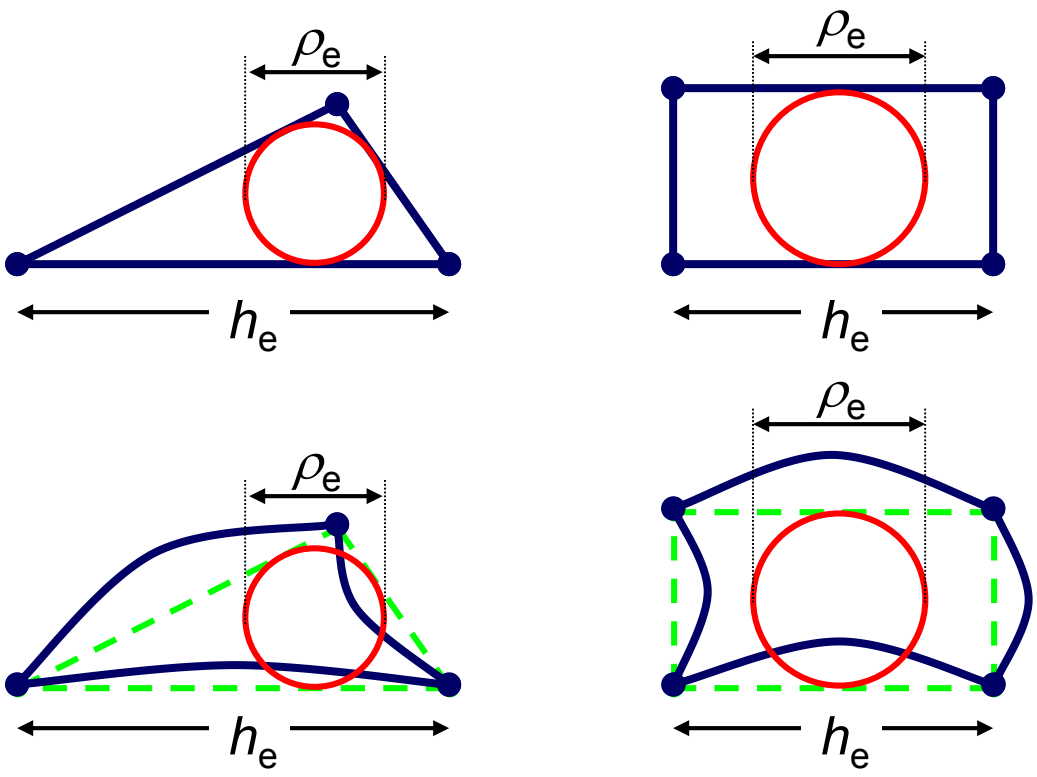
Elements with curved boundary can be adopted in 2D, to better represent curved shapes.



Conformal (curved) elements can be also adopted in 3D, to better represent shapes with non-planar interfaces.

The accuracy of the method increases if the shape of the elements is "regular" (equilateral triangles, squares, tetrahedra with equilateral faces, cubes, …).

It is convenient to introduce a measure of the **element distortion** (σ_e):



$$\sigma_e = h_e/\rho_e$$

h_e max distance of two element vertices

ρ_e diameter of the max inscribed circle (2D) or sphere (3D)

Definition of the Interpolating Functions

INTERPOLATION FUNCTIONS

The second step in the implementation of the FEM is the definition of the interpolating functions, used to approximate the unknown function inside each element.

Interpolating functions are typically linear or quadratic functions, sometimes higher order polynomial functions.

The unknown function ϕ^e in the e-th element is express as:

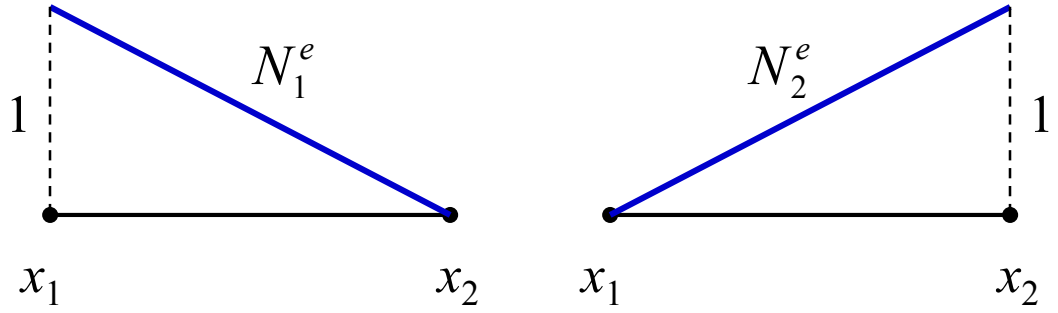
$$\phi^e = \sum_{j=1}^{n} N_j^e \, \phi_j^e$$

where n is the number of element nodes, ϕ_j^e is the value of the unknown function in the j-th node, and N_j^e is the interpolating function (or expansion function or basis function)

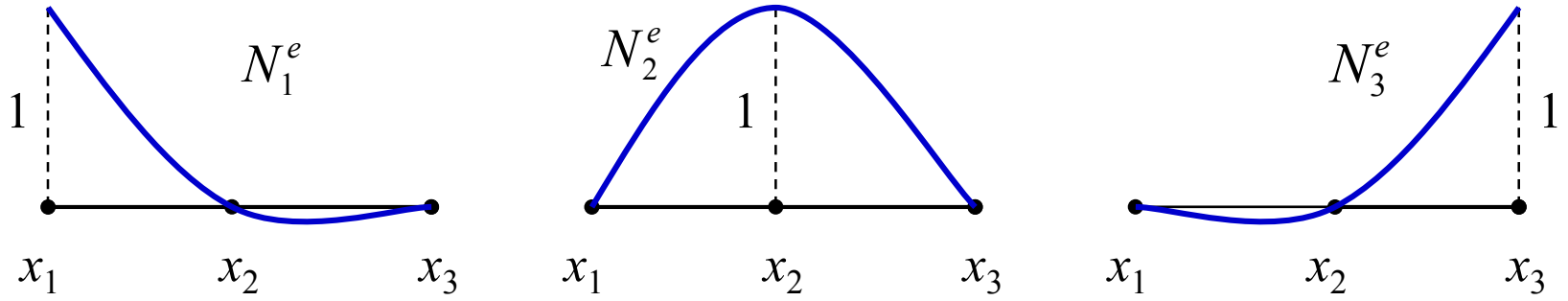
NOTE: the interpolating function N_j^e vanishes out of the e-th element.

1D Interpolation Functions

LINEAR FUNCTIONS

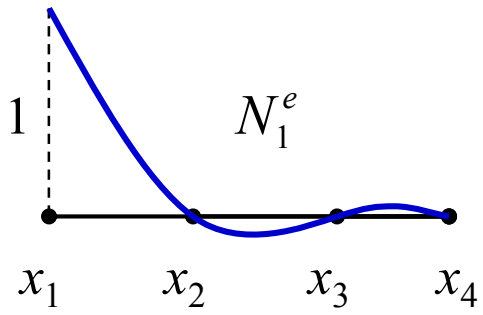


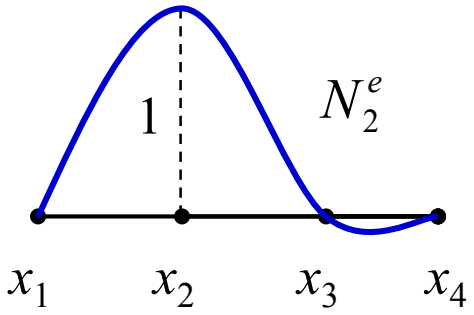
QUADRATIC FUNCTIONS

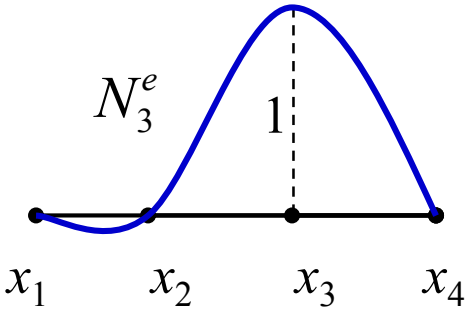


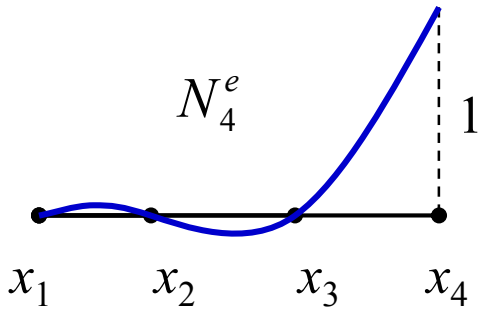
1D Interpolation Functions

CUBIC FUNCTIONS









APPROXIMATION ERROR

For a given number of nodes, the error decreases when increasing the order of the interpolating functions.

The plot refers to a 1D case (field reflected by a stratified medium, with incident plane wave).

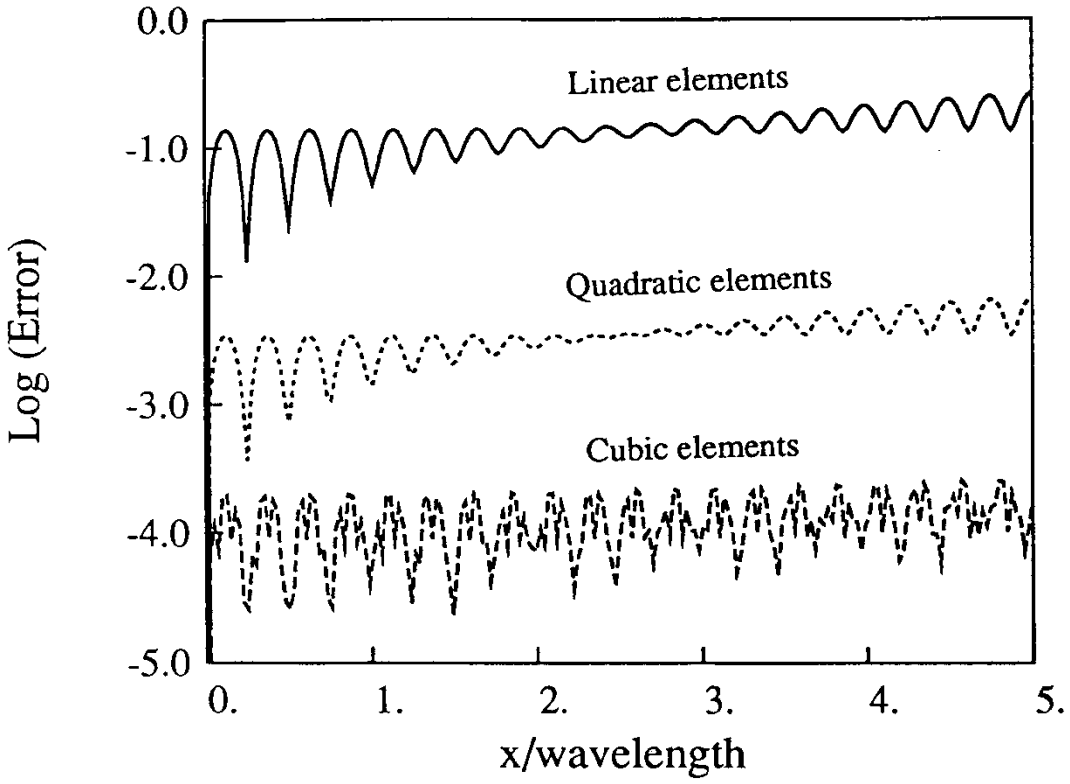


FIGURE 3.15 Comparison of the error in the finite element solution obtained with linear, quadratic, and cubic elements, all using 199 nodes.

APPROXIMATION ERROR

This plot shows the error versus the number of nodes for different types of interpolating functions.

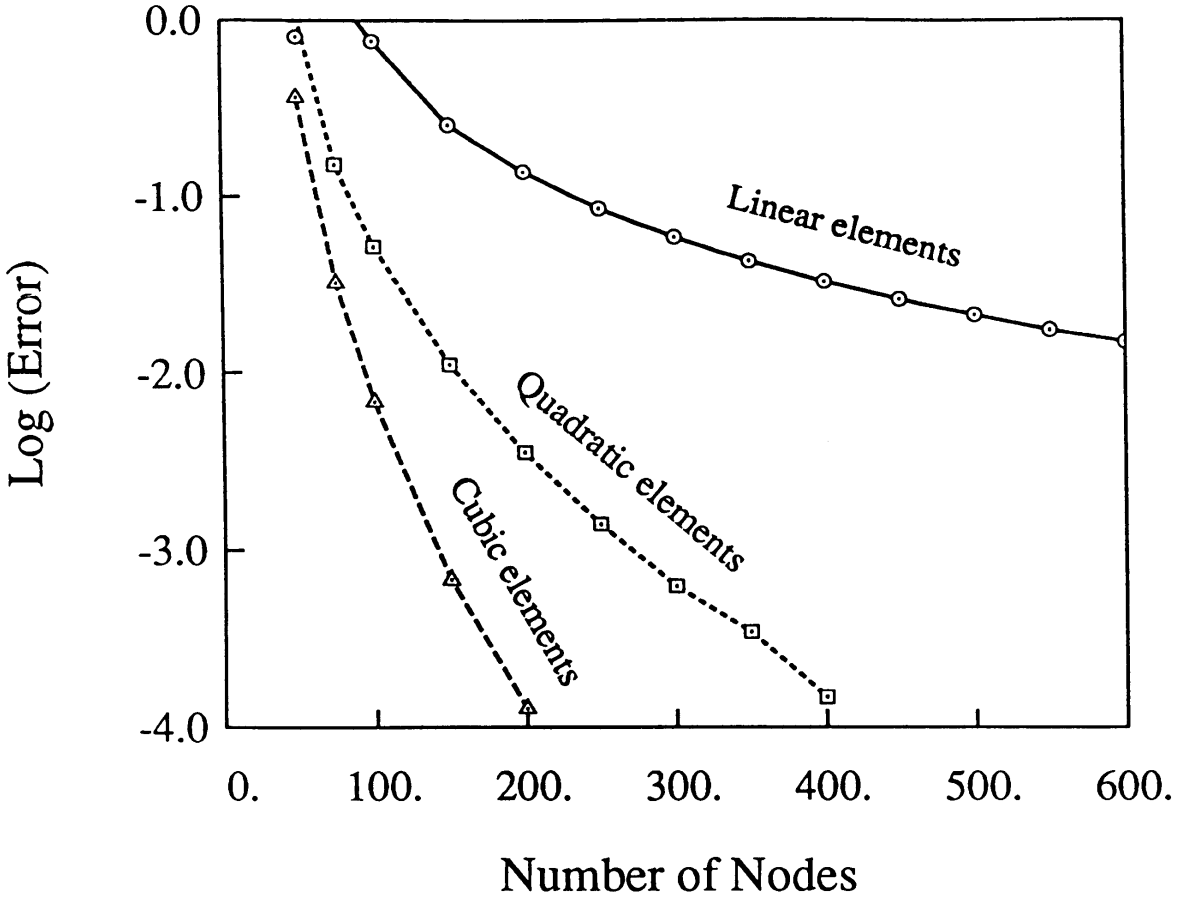
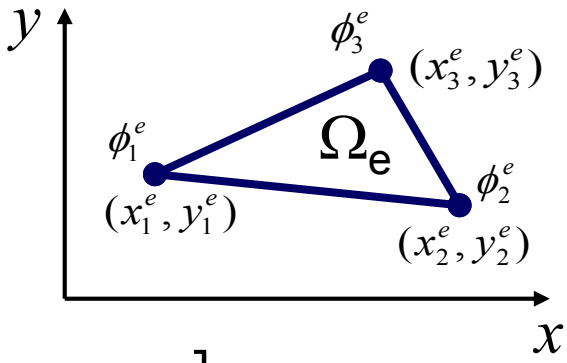


FIGURE 3.16 Average error of the finite element solution versus the number of nodes.

2D Interpolation Functions

The unknown function in the e-th element is expressed as

$$\phi^e(x,y) = \sum_{i=1}^{3} \alpha_i^e(x,y)\phi_i^e$$

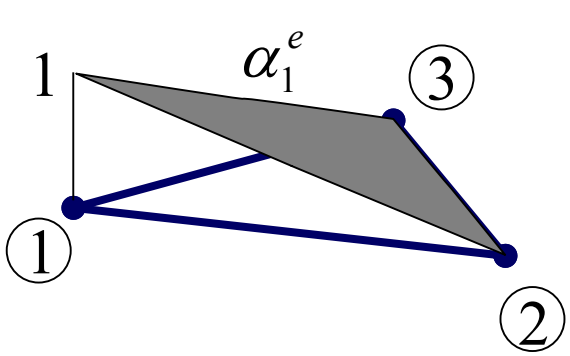


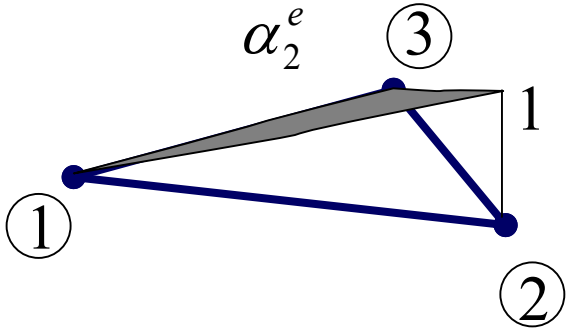
where

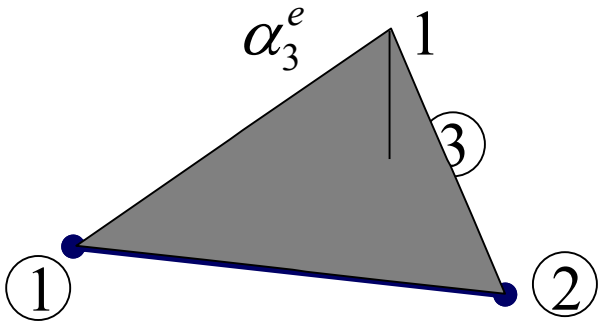
$$\alpha_1^e(x,y) = \frac{1}{2A}\left[(x_2^e y_3^e - x_3^e y_2^e) + (y_2^e - y_3^e)x + (x_3^e - x_2^e)y\right]$$

$$\alpha_2^e(x,y) = \frac{1}{2A}\left[(x_3^e y_1^e - x_1^e y_3^e) + (y_3^e - y_1^e)x + (x_1^e - x_3^e)y\right]$$

$$\alpha_3^e(x,y) = \frac{1}{2A}\left[(x_1^e y_2^e - x_2^e y_1^e) + (y_1^e - y_2^e)x + (x_2^e - x_1^e)y\right]$$



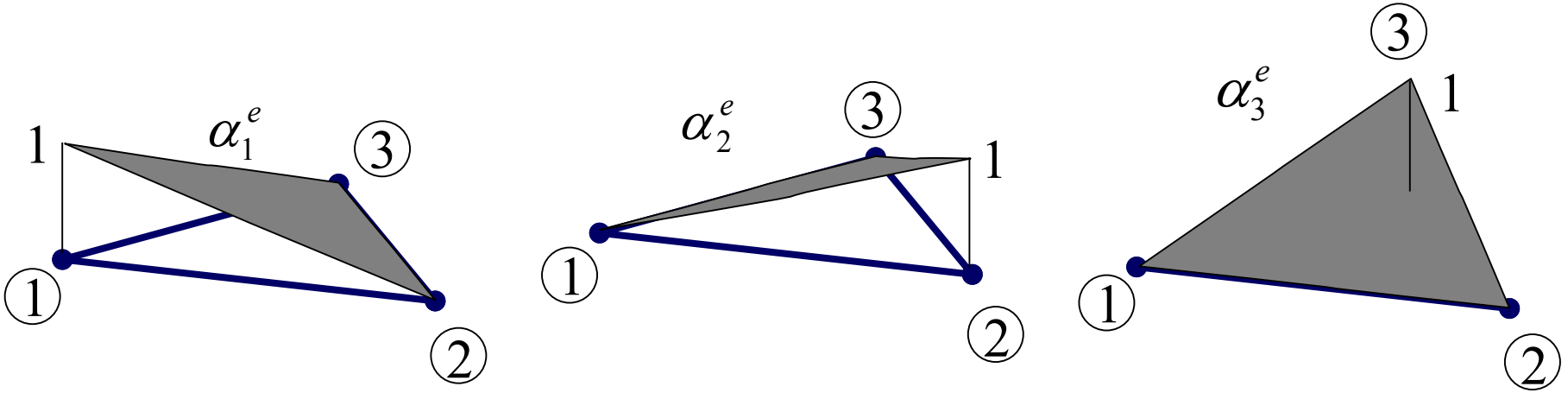




2D Interpolation Functions

These interpolating functions are defined Lagrangian functions and they satisfy the following conditions:

$$\sum_{i=1}^{3} \alpha_i^e(x,y) = 1 \qquad \alpha_i^e(x_j^e, y_j^e) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$



NOTE: this feature of the interpolating functions guarantees the continuity of the unknown functions between adjacent elements.

3D Vector Functions

In 3D vector problems, scalar interpolation functions can be adopted:

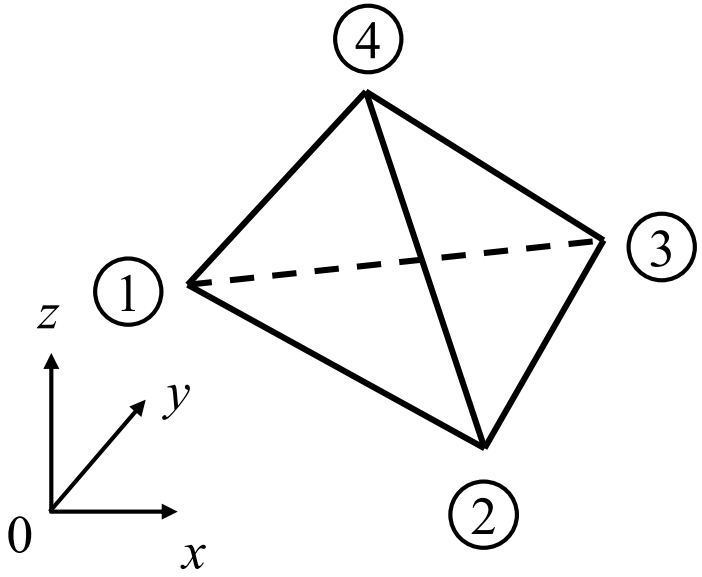
$$\mathbf{E}^e(x,y,z) = \hat{\mathbf{x}}\, E_x^e(x,y,z) + \hat{\mathbf{y}}\, E_y^e(x,y,z) + \hat{\mathbf{z}}\, E_z^e(x,y,z)$$

where

$$E_x^e(x,y,z) = \sum_{i=1}^{4} \alpha_i^e(x,y,z) E_{x,i}^e$$

$$E_y^e(x,y,z) = \sum_{i=1}^{4} \alpha_i^e(x,y,z) E_{y,i}^e$$

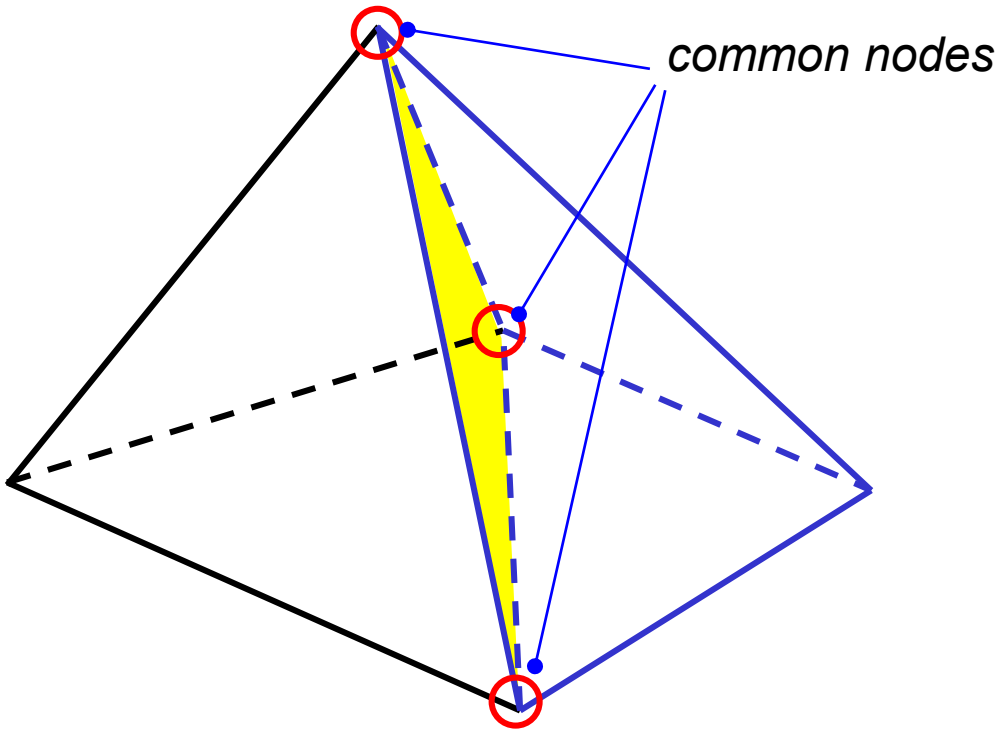
$$E_z^e(x,y,z) = \sum_{i=1}^{4} \alpha_i^e(x,y,z) E_{z,i}^e$$



This choice of the interpolation functions automatically enforces the inter-elements continuity of the function.

BOUNDARY CONDITION AT INTERFACES

The condition $\mathbf{n} \times \mathbf{E}^{+} = \mathbf{n} \times \mathbf{E}^{-}$ at the interface between adjacent tetrahedra filled with different materials is automatically guaranteed, since the unknown function has the same value at common nodes.



The common nodes impose that also the normal components of the electric fields are continuous, thus violating the boundary condition at the interface.

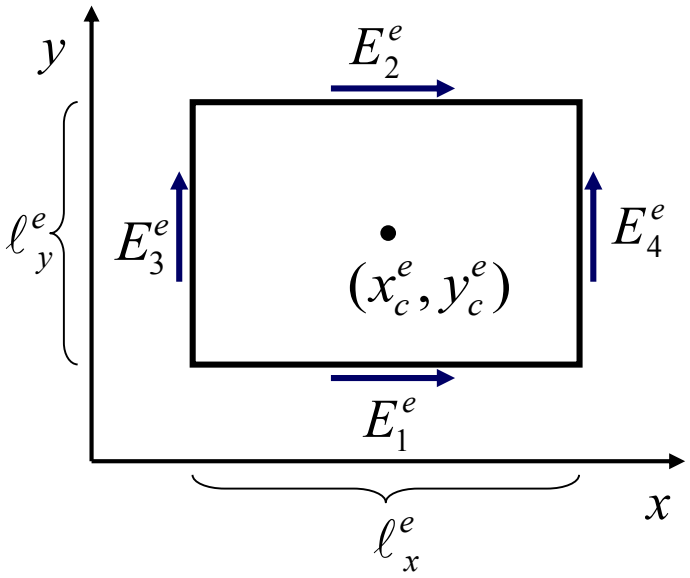
The problem of the boundary condition at the interfaces and of the spurious solution can be solved by adopting different interpolation functions, named "**edge element**". In the case of rectangular domains:

$$\mathbf{E}^e(x,y) = \sum_{i=1}^{4} \boldsymbol{\alpha}_i^e(x,y) E_i^e$$

where

$$\boldsymbol{\alpha}_1^e = \left(\frac{1}{2} + \frac{y_c^e - y}{\ell_y^e}\right)\hat{\mathbf{x}} \qquad \boldsymbol{\alpha}_2^e = \left(\frac{1}{2} + \frac{y - y_c^e}{\ell_y^e}\right)\hat{\mathbf{x}}$$

$$\boldsymbol{\alpha}_3^e = \left(\frac{1}{2} + \frac{x_c^e - x}{\ell_x^e}\right)\hat{\mathbf{y}} \qquad \boldsymbol{\alpha}_4^e = \left(\frac{1}{2} + \frac{x - x_c^e}{\ell_x^e}\right)\hat{\mathbf{y}}$$



This approach easily permits to guarantee the continuity of the tangential component of the E field and it can be shown that $\nabla \cdot \boldsymbol{\alpha}_i^e = 0$ in each element.

FORMULATION OF THE MATRIX EQUATION

FORMULATION OF THE LINEAR SYSTEM / 1

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The formulation of the matrix equation is the most important step in the implementation of the FEM, and it can be based either on the Rayleigh-Ritz method or on the Galerkin method.

A. Formulation based on the Rayleigh-Ritz method

We consider the linear functional equation

$$L\phi = f \quad \text{in } \Omega$$

where L is a self-adjoint differential operator, f is a known function and ϕ is the unknown function to determine. The functional F associate to the differential operator L with the boundary condition is:

$$F(\phi) = \frac{1}{2}\langle L\phi, \phi \rangle - \frac{1}{2}\langle \phi, f \rangle - \frac{1}{2}\langle f, \phi \rangle$$

and the inner product is defined as $\langle \phi, \psi \rangle = \int_{\Omega} \phi \psi^* d\Omega$.

FORMULATION OF THE LINEAR SYSTEM / 2

The functional F can be expressed as the sum of the contributions F^e of the single elements:

$$F(\phi) = \sum_{e=1}^{M} F^e(\phi^e)$$

where M is the number of elements in the entire domain. Therefore:

$$F^e(\phi^e) = \frac{1}{2}\int_{\Omega^e} \phi^e L \phi^e \, d\Omega - \int_{\Omega^e} f \, \phi^e \, d\Omega$$

As the unknown function ϕ^e is expressed as a linear combination of interpolating functions

$$\phi^e = \sum_{j=1}^{n} N_j^e \, \phi_j^e = [N^e]^T [\phi^e] = [\phi^e]^T [N^e]$$

the functional F^e results:

$$F^e(\phi^e) = \frac{1}{2}[\phi^e]^T \underbrace{\left\{ \int_{\Omega^e} [N^e] \, L \, [N^e]^T \, d\Omega \right\}}_{[K^e]} [\phi^e] - [\phi^e]^T \underbrace{\int_{\Omega^e} f \, [N^e] \, d\Omega}_{[b^e]}$$

NOTE: matrix $[K^e]$ is symmetric as the operator is self-adjoint.

FORMULATION OF THE LINEAR SYSTEM / 3

The functional F can be expressed as the sum of the contributions F^e:

$$F(\phi) = \sum_{e=1}^{M} F^e(\phi^e) = \sum_{e=1}^{M} \left(\frac{1}{2} [\phi^e]^T [K^e] [\phi^e] - [\phi^e]^T [b^e] \right)$$

By adopting a global numbering of the nodes:

$$F(\phi) = \frac{1}{2} [\phi]^T [K] [\phi] - [\phi]^T [b]$$

where $[K]$ is a $N \times N$ matrix (N=number of nodes), $[\phi]$ is the vector with the N coefficients of the interpolating functions, $[b]$ is a vector with N elements.

The matrix equation is obtained by setting to zero the partial derivatives of the functional F with respect to coefficients ϕ_i:

$$\frac{\partial F}{\partial \phi_i} = \frac{1}{2} \sum_{j=1}^{N} (K_{ij} + K_{ji}) \phi_j - b_i = \sum_{j=1}^{N} K_{ij} \phi_j - b_i = 0 \qquad i = 1, 2, .., N$$

In matrix form, the linear system results:

$$\boxed{[K][\phi] = [b]}$$

B. Formulation based on the Galerkin method

We consider the linear functional equation

$$L\phi = f \quad \text{in } \Omega$$

where L is a differential operator (non necessarily self-adjoint), f is a known function and ϕ is the unknown function to determine.

The unknown function ϕ^e in the element Ω^e is expressed as a linear combination of interpolating functions

$$\phi^e = \sum_{j=1}^{n} N_j^e \, \phi_j^e = [N^e]^T [\phi^e] = [\phi^e]^T [N^e]$$

where N_j^e represent the interpolating functions, and ϕ_j^e are the values of the unknown function in the nodes of the e-th element.

By replacing this representation of ϕ^e in the differential equation and applying the test with identical test functions, it results:

$$\left(\int_{\Omega^e} N_i^e \, L\,[N^e]^T \, d\Omega \right) [\phi^e] = \int_{\Omega^e} N_i^e \, f \, d\Omega \qquad i = 1, 2, .., N$$

In matrix form it results:

$$[K^e][\phi^e] = [b^e]$$

If the operator L is self-adjoint, matrix $[K^e]$ is symmetric.

By assembling the matrices of the single elements as in the previous case, the final matrix equation is:

$$\boxed{[K][\phi] = [b]}$$

Solution of the Matrix Equation

SOLUTION OF THE LINEAR SYSTEM

The solution of the matrix equation in the final step in the FEM analysis. The resulting matrix equation can assume two forms.

A. Deterministic problem (with excitation)

In the case of scattering or radiation problems, where there is an excitation of the system, the resulting matrix equation results:

$$[K][\phi] = [b]$$

B. Eigenvalue problem (no excitation)

In the calculation of the proper modes of a waveguide or a cavity, there is no excitation. In the case the resulting matrix equation is homogeneous ($[b]=0$). In this case matrix $[K]$ can be expressed as $[K]=[A]-\lambda[B]$, where λ represent the problem eigenvalue, and the matrix equation results:

$$[A][\phi] = \lambda[B][\phi]$$