Lecture 12

APPLICATIONS OF THE FINITE ELEMENT METHOD (FEM)

OUTLINE

University of Pavia

In this lecture, two applications of the FEM are presented and discussed:

1. FEM solution of the Laplace equation in 1D case

2. FEM solution of the wave equation in 3D case

APPLICATION OF THE FINITE ELEMENT METHOD TO 1D LAPLACE EQUATION

LAPLACE EQUATION / 1

In this section, the FEM is applied to the solution of the Laplace equation in the one-dimensional case:

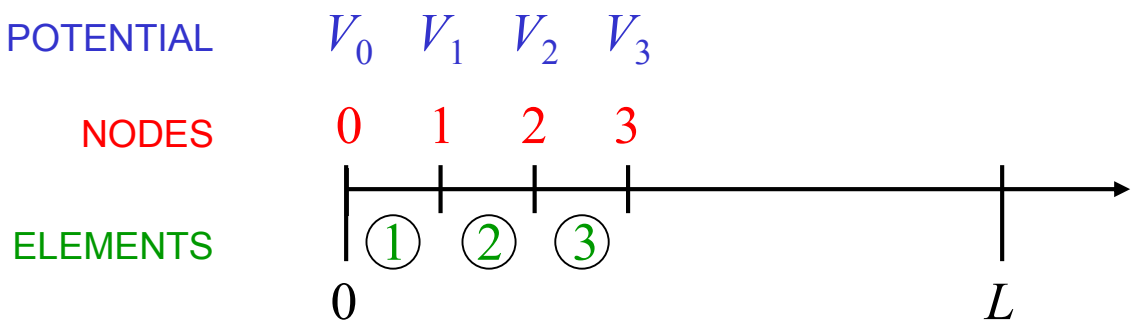
$$\nabla^2 V(x) = 0 \qquad (0 \le x \le L)$$

with boundary conditions:

$$V(0)=a \qquad V(L)=b$$

1. DOMAIN DISCRETIZATION

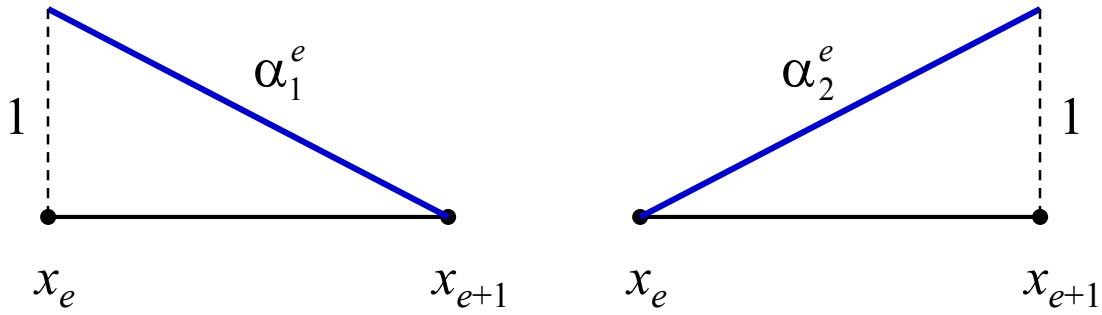
The 1D domain $0 \le x \le L$ is subdivided in N elements, with length $\Delta x = L/N$.



2. INTERPOLATION FUNCTIONS

In each element, the unknown function $V(x)$ is represented as the combination of two linear interpolation functions

$$V^e(x) = \sum_{i=1}^{2} V_i^e \alpha_i^e(x)$$



where the interpolation functions are defined as

$$\alpha_1^e(x) = \frac{x_{e+1} - x}{\Delta x} \qquad \alpha_2^e(x) = \frac{x - x_e}{\Delta x}$$

and

$$\Delta x = x_{e+1} - x_e$$

3. FORMULATION OF THE MATRIX PROBLEM

The functional associated to the Laplace equation in the e-th element is:

$$F^e = \frac{1}{2}\int_{x_e}^{x_{e+1}} \varepsilon_e \left|\nabla V^e\right|^2 dx$$

where ε_e represents la relative dielectric permittivity in the e-th element. By replacing the expression of V^e in the functional, it results:

$$F^e = \frac{1}{2}\int_{x_e}^{x_{e+1}} \varepsilon_e \left|\nabla V^e\right|^2 dx = \frac{1}{2}\int_{x_e}^{x_{e+1}} \varepsilon_e \left[\sum_{i=1}^{2} V_i^e \nabla\alpha_i^e(x)\right]\cdot\left[\sum_{j=1}^{2} V_j^e \nabla\alpha_j^e(x)\right] dx =$$

$$= \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} V_i^e \underbrace{\int_{x_e}^{x_{e+1}} \varepsilon_e \nabla\alpha_i^e \cdot \nabla\alpha_j^e \, dx}_{C_{ij}^e} V_j^e = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} V_i^e \, C_{ij}^e \, V_j^e$$

where

$$C_{11}^e = \varepsilon_e \nabla\alpha_1^e \cdot \nabla\alpha_1^e \, \Delta x = \frac{\varepsilon_e}{\Delta x} = C_{22}^e \qquad C_{12}^e = C_{21}^e = -\frac{\varepsilon_e}{\Delta x}$$

LAPLACE EQUATION / 4

The global functional is obtained by combining the contribution of all elements

$$F = \sum_{e=1}^{N} F^e = \frac{1}{2}[V]^T [C][V]$$

where matrix $[C]$ and vector $[V]$ are defined after the global numbering of the nodes:

$$[V] = \begin{bmatrix} V_0 \\ V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix} \quad [C] = \begin{bmatrix} C_{11}^1 & C_{12}^1 & 0 & 0 & \\ C_{21}^1 & C_{22}^1 + C_{11}^2 & C_{12}^2 & 0 & \cdots \\ 0 & C_{21}^2 & C_{22}^2 + C_{11}^3 & C_{12}^3 & \\ 0 & 0 & C_{21}^3 & C_{22}^3 + C_{11}^4 & \\ & \vdots & & & \ddots \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} & 0 & 0 & \\ C_{10} & C_{11} & C_{12} & 0 & \cdots \\ 0 & C_{21} & C_{22} & C_{23} & \\ 0 & 0 & C_{32} & C_{33} & \\ & \vdots & & & \ddots \end{bmatrix}$$

LAPLACE EQUATION / 5

The problem is formulated by minimizing the functional with respect to the variables $V_1 \ldots V_N$

$$\frac{\partial F}{\partial V_k} = 0 \qquad (k=0 \,..\, N)$$

After incorporating the boundary conditions (which remove two variables)

$$V_0 = a \qquad V_N = b$$

the final matrix problem results:

$$\begin{bmatrix} C_{11} & C_{12} & 0 & 0 & & \\ C_{21} & C_{22} & C_{23} & 0 & & \\ 0 & C_{32} & C_{33} & & & \\ 0 & 0 & & \ddots & & \\ & & & & C_{N-1,N-1} & \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{N-1} \end{bmatrix} = \begin{bmatrix} -C_{10}a \\ 0 \\ 0 \\ \vdots \\ -C_{N-1,N}b \end{bmatrix}$$

4. SOLUTION OF THE MATRIX PROBLEM

The resulting matrix problem has dimension $(N-1)\times(N-1)$, where N is the number of elements. The solution of the matrix problem provides the value of the potential V in all internal nodes.

NOTE – It is noted that, with a suitable numbering of the nodes, the matrix problem is a band matrix, where only three terms per row differ from zero, clustered near the main diagonal.

This peculiarity of the problem can be exploited to reduce computing time and memory allocation space.

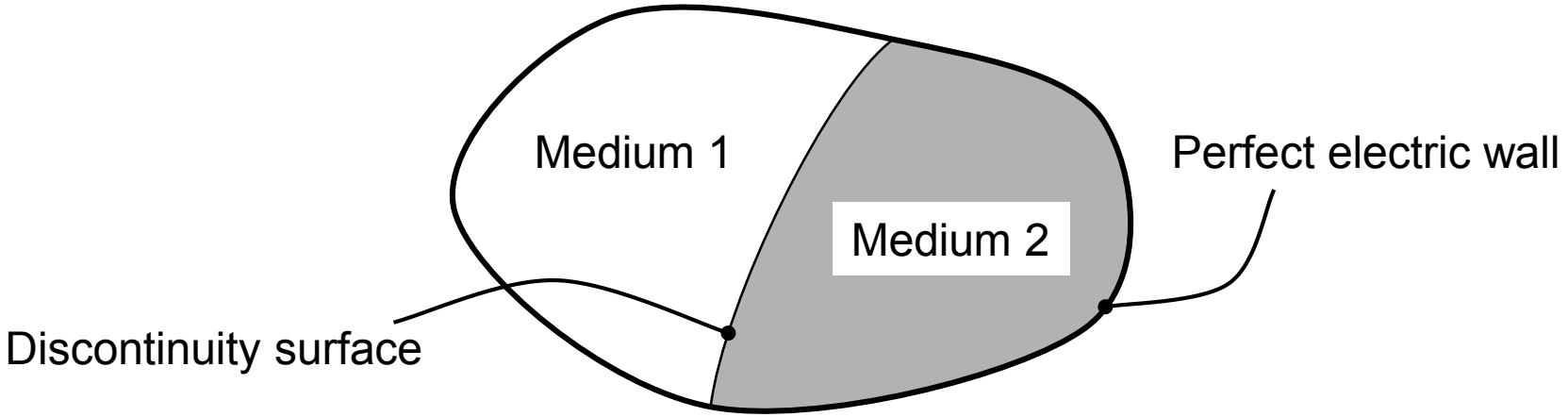
APPLICATION OF THE FINITE ELEMENT METHOD TO THE 3D WAVE EQUATION

3D Wave Equation / 1

We consider the wave equation in the 3D case:

$$\nabla \times \left(\frac{1}{\mu_r} \nabla \times \mathbf{E} \right) - k_0^2 \varepsilon_c \mathbf{E} = -jk_0 Z_0 \mathbf{J}$$

In a domain filled with two different materials and with perfect electric wall condition ($\mathbf{n} \times \mathbf{E} = 0$) at the boundary of the domain and field continuity condition ($\mathbf{n} \times \mathbf{E}^+ = \mathbf{n} \times \mathbf{E}^-$) at the discontinuity surface.



3D Wave Equation / 2

In this case, the functional to minimize is the following:

$$F(\mathbf{E}) = \frac{1}{2}\iiint_V \left[\frac{1}{\mu_r}(\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{E}) - k_0^2 \varepsilon_c \mathbf{E} \cdot \mathbf{E} \right] dV + jk_0 Z_0 \iiint_V \mathbf{E} \cdot \mathbf{J}\, dV$$

with the boundary conditions:

$$\mathbf{n} \times \mathbf{E} = 0 \qquad \text{at the boundary}$$

$$\mathbf{n} \times \mathbf{E}^+ = \mathbf{n} \times \mathbf{E}^- \qquad \text{at the interface}$$

3D Vector Functions

In 3D vector problems, scalar interpolation functions can be adopted:

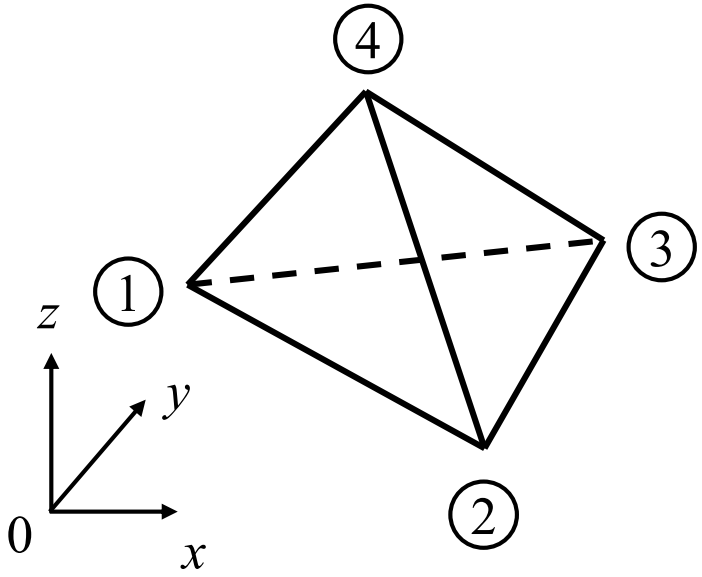
$$\mathbf{E}^e(x,y,z) = \hat{\mathbf{x}}\, E_x^e(x,y,z) + \hat{\mathbf{y}}\, E_y^e(x,y,z) + \hat{\mathbf{z}}\, E_z^e(x,y,z)$$

where

$$E_x^e(x,y,z) = \sum_{i=1}^{4} \alpha_i^e(x,y,z) E_{x,i}^e$$

$$E_y^e(x,y,z) = \sum_{i=1}^{4} \alpha_i^e(x,y,z) E_{y,i}^e$$

$$E_z^e(x,y,z) = \sum_{i=1}^{4} \alpha_i^e(x,y,z) E_{z,i}^e$$



This choice of the interpolation functions automatically enforces the inter-elements continuity of the function.

BOUNDARY CONDITIONS

The perfect electric wall condition ($\mathbf{n} \times \mathbf{E} = 0$) can be formulated for the three Cartesian components x, y, and z:

$$n_y E_z - n_z E_y = 0$$
$$n_z E_x - n_x E_z = 0$$
$$n_x E_y - n_y E_x = 0$$

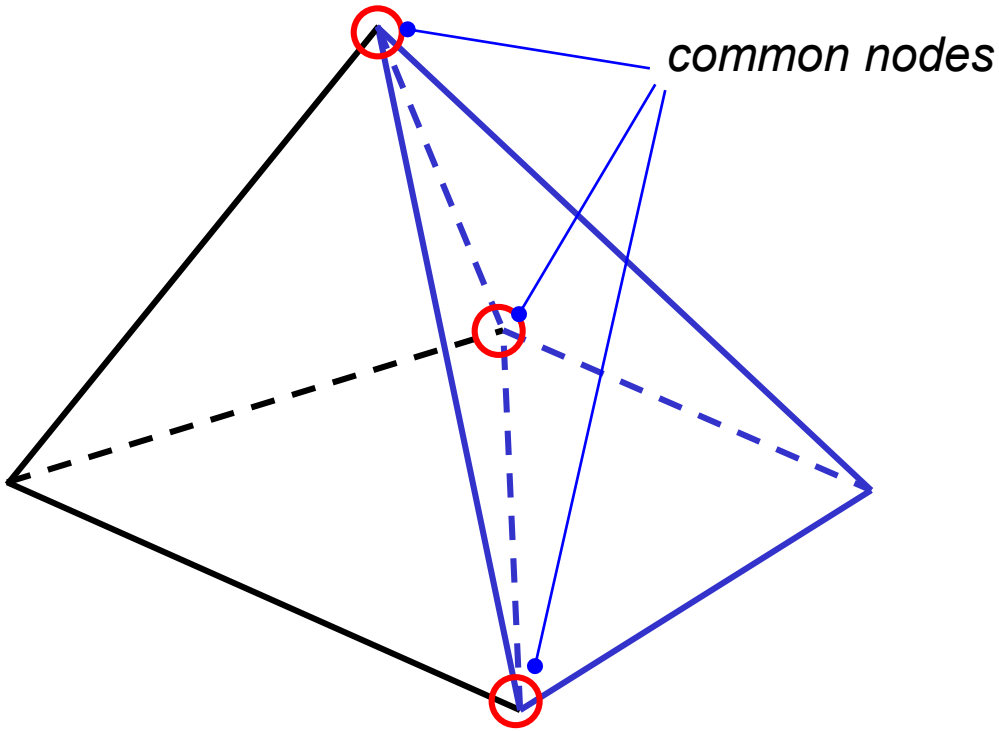
where n_x, n_y, n_z are defined as $\mathbf{n} = \hat{\mathbf{x}} n_x + \hat{\mathbf{y}} n_y + \hat{\mathbf{z}} n_z$.

It is possible to express two components as a function of the third one. For instance, if $n_z \neq 0$, it results:

$$E_x = \frac{n_x}{n_z} E_z \qquad E_y = \frac{n_y}{n_z} E_z$$

BOUNDARY CONDITION AT INTERFACES / 1

The condition $\mathbf{n} \times \mathbf{E}^{+} = \mathbf{n} \times \mathbf{E}^{-}$ at the interface between adjacent tetrahedra filled with different materials is automatically guaranteed, since the unknown function has the same value at common nodes.



BOUNDARY CONDITION AT INTERFACES / 2

The common nodes impose that also the normal components of the electric fields are continuous, thus violating the boundary condition at the interface

$$\mathbf{n}\cdot(\varepsilon_c^{+}\mathbf{E}^{+}) = \mathbf{n}\cdot(\varepsilon_c^{-}\mathbf{E}^{-})$$

However, this condition is a natural condition of the problem and therefore it tends to be automatically satisfied by the field.

It can be shown that a finer mesh near the interface allows to achieve a very rapid variation of the field, that emulates the discontinuity effect.

BOUNDARY CONDITION AT INTERFACES / 3

A more rigorous approach consists in duplicating the nodes at the interface. In this way, there are enough degrees of freedom to explicitly impose the conditions:

$$\mathbf{n}\times\mathbf{E}^{+}=\mathbf{n}\times\mathbf{E}^{-} \qquad \mathbf{n}\cdot(\varepsilon_c^{+}\mathbf{E}^{+})=\mathbf{n}\cdot(\varepsilon_c^{-}\mathbf{E}^{-})$$

It results:

$$\begin{bmatrix} E_x^+ \\ E_y^+ \\ E_z^+ \end{bmatrix} = \begin{bmatrix} n_x^2 e+1 & n_x n_y e & n_x n_z e \\ n_x n_y e & n_y^2 e+1 & n_y n_z e \\ n_x n_z e & n_y n_z e & n_z^2 e+1 \end{bmatrix} \begin{bmatrix} E_x^- \\ E_y^- \\ E_z^- \end{bmatrix}$$

with:

$$e=\frac{\varepsilon_c^-}{\varepsilon_c^+}-1$$

SPURIOUS SOLUTIONS

The application of the FEM can sometimes give erroneous solutions, because the resulting electric field does not satisfy the divergence equation

$$\nabla \cdot (\varepsilon_c \mathbf{E}) = -\frac{\nabla \cdot \mathbf{J}}{j\omega}$$

even though this equation derives from the wave equation

$$\nabla \times \left(\frac{1}{\mu_r} \nabla \times \mathbf{E} \right) - k_0^2 \varepsilon_c \mathbf{E} = -jk_0 Z_0 \mathbf{J}$$

simply by taking the divergence of both sides.

This is due to the regularity of the interpolation functions, which are C_0 across interfaces (C_1 regularity would be required).

SPURIOUS SOLUTIONS

Different solutions have been proposed to solve the spurious solutions:

1. using C_1 interpolation functions (too complicate, unpractical)
2. formulating the problem for the **H** field, and then obtaining the **E** field as

$$\mathbf{E} = \frac{1}{j\omega\varepsilon_c}(\nabla \times \mathbf{H} - \mathbf{J})$$

3. adding a penalty term in the functional, whose minimization enforces the divergence condition

$$F(\mathbf{E}) =$$

$$= \frac{1}{2}\iiint_V \left[\frac{1}{\mu_r}(\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{E}) + \frac{s}{\mu_r \varepsilon_c^2}\left(\nabla \cdot (\varepsilon_c \mathbf{E}) + \frac{\nabla \cdot \mathbf{J}}{j\omega\varepsilon_0} \right)^2 - k_0^2 \varepsilon_c \mathbf{E} \cdot \mathbf{E} \right] dV$$

$$+ jk_0 Z_0 \iiint_V \mathbf{E} \cdot \mathbf{J}\, dV$$

FAST SWEEP TECHNIQUES

Frequency domain solvers typically require to discretize the problem, build a matrix, and solve the matrix problem at each frequency of interest.

Fast sweep methods attempts to find a rational polynomial that describes the solution behavior with a minimum set of computer frequency points. They are typically based on:

- Asymptotic waveform evaluation (AWE)
- Padé via Lanczos method (PVL)
- Adaptive Lanczos-Padé sweep (ALPS)

Nowaday these methods are quite reliable and they typically provide a substantial reduction in computing time.