Lecture 2

FDTD: THE FINITE DIFFERENCE TIME DOMAIN METHOD

INTRODUCTION

The finite-difference time-domain (FDTD) method is a classical technique for the modeling of circuits, antennas and scattering problems.

The FDTD method is easy to implement and very flexible: it can apply to the modeling of complex regions, with inhomogeneous medium and the mixed/time-dependent boundary conditions. The solution is calculated in the time domain, thus permitting the study of transient behavior with pulsed source.

Originally proposed in the 1920's by A. Thom to solve hydrodynamic equations, it was applied to the electromagnetic modeling by K. Yee in 1966 for the first time.

INTRODUCTION

The FDTD method is based on the approximation of the partial derivatives appearing in the differential equations as finite differences.

The implementation of the FDTD method involves three steps:

1. definition of a grid of nodes in the solution region
2. approximation of the partial derivatives as finite differences
3. iterative solution of the equation starting from the initial conditions

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Reference books:

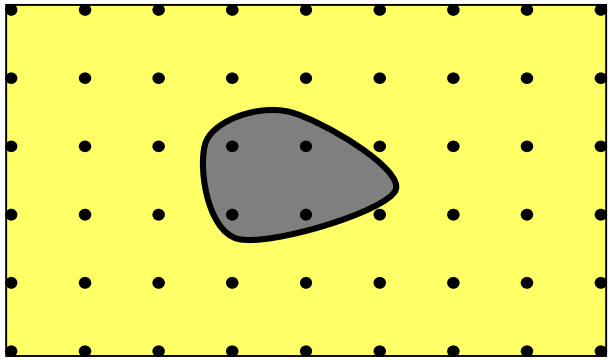
1. M.N.O. Sadiku, *Numerical techniques in Electromagnetics*, CRC Press, 2000.
2. A. Taflove and S.C. Hagness, *Computational Electrodynamics: The Finite-Difference Time-Domain Method*, Artech House, 2005.

DEFINITION OF THE GRID / 1

The application of the FDTD method requires the definition of a grid of nodes in the whole investigation domain.

The spacing of the grid nodes is related to the **geometry** and to the **variation of the electromagnetic field**.

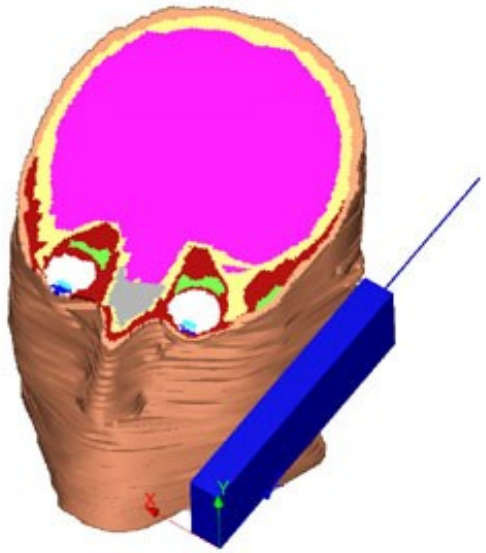
In general accurate results are obtained when the grid spacing is around $\lambda/20$ at the maximum frequency of interest.



DEFINITION OF THE GRID / 2

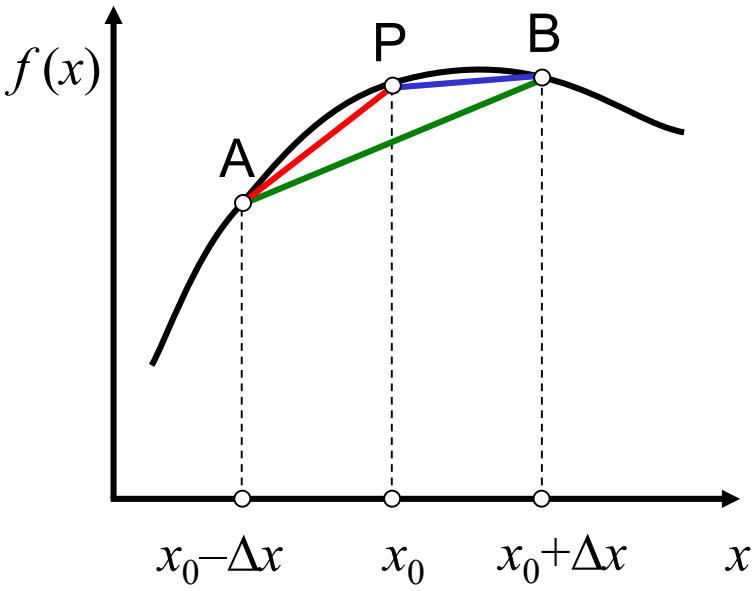
In some cases, the **sub-gridding technique** is adopted, to reduce the number of nodes.

It results very convenient in the modeling of wire antennas and PEC structures to arbitrarily oriented PEC geometries. This enables a much improved representation compared to common staircase modeling.



FINITE DIFFERENCE SCHEMES / 1

Given a function $f(x)$, its derivative in P can be approximated in different ways:



- FORWARD DIFFERENCE:

$$f'(x_0) \cong \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

- BACKWARD DIFFERENCE:

$$f'(x_0) \cong \frac{f(x_0)-f(x_0-\Delta x)}{\Delta x}$$

- CENTRAL DIFFERENCE:

$$f'(x_0) \cong \frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x}$$

Similarly, the second derivative can be approximated as:

$$f''(x_0) \cong \frac{f'(x_0 + \Delta x/2) - f'(x_0 - \Delta x/2)}{\Delta x}$$

$$\cong \frac{1}{\Delta x}\left(\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} - \frac{f(x_0) - f(x_0 - \Delta x)}{\Delta x} \right)$$

$$\cong \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{(\Delta x)^2}$$

By using this technique, the derivatives f' and f'' can be approximated in terms of the value of the function in a **discrete set of points**.

FINITE DIFFERENCE SCHEMES / 3

A more systematic approach to achieve the same result is based on the Taylor's series:

$$f(x_0 \pm \Delta x) = f(x_0) \pm \Delta x\, f'(x_0) + \frac{1}{2!}(\Delta x)^2\, f''(x_0) \pm \frac{1}{3!}(\Delta x)^3\, f'''(x_0) + \cdots$$

Upon subtracting the expansion with sign + and the one with sign –, after dividing by $2\Delta x$, it results:

$$f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} + O(\Delta x^2)$$

truncation error

Similarly, upon adding the expansion with sign + and the one with sign –, after dividing by $(\Delta x)^2$, it results:

$$f''(x_0) = \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{(\Delta x)^2} + O(\Delta x^2)$$

It is noted that forward and backward difference formulas have truncation error of $O(\Delta x)$.

ITERATIVE SOLUTION / 1

A simple example is useful to introduce the concept of the iterative solution.

Let us consider the following partial differential equation

$$k\frac{\partial \Phi}{\partial t} = \frac{\partial^2 \Phi}{\partial x^2}$$

(where *k* is constant).

We define a space grid with step Δx and a time step Δt (thus resulting $x = i\,\Delta x$ *and* $t = j\,\Delta t$, with $i, j = 0,1,\ldots$)

We approximate the time derivative with the forward difference and the space derivative with the central difference*:*

$$k\frac{\Phi(i, j+1) - \Phi(i, j)}{\Delta t} = \frac{\Phi(i+1, j) - 2\Phi(i, j) + \Phi(i-1, j)}{(\Delta x)^2}$$

It is observed that $\Phi(i,j)$ is a compact form for representing $\Phi(i\,\Delta x,\ j\,\Delta t)$

ITERATIVE SOLUTION / 2

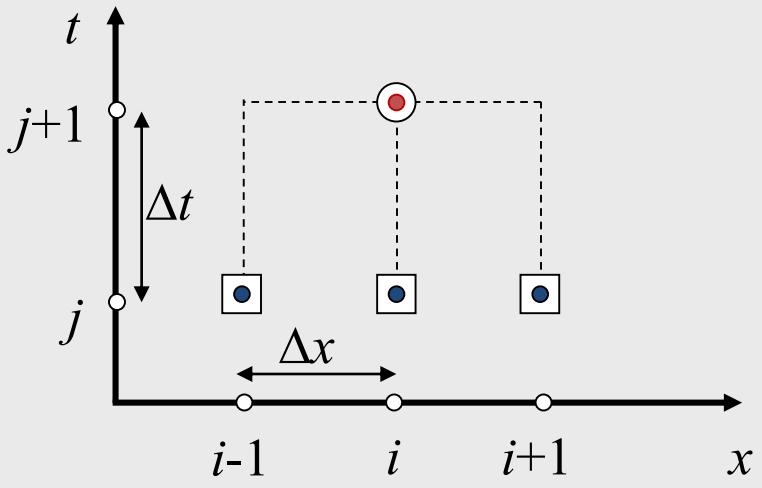
After re-ordering the equation, we obtain:

$$\Phi(i, j+1) = r\,\Phi(i+1, j) + (1-2r)\,\Phi(i, j) + r\,\Phi(i-1, j) \qquad \text{(with } r = \frac{\Delta t}{k(\Delta x)^2}\text{).}$$

This expression represents an **explicit formula** to determine $\Phi(x,t+\Delta t)$ in terms of $\Phi(x,t)$.

COMPUTATIONAL SCHEME

- value to determine
- known values of Φ



This explicit formula is stable only for certain values of *r*.

$$0 < r \leq \frac{1}{2}$$

STABILITY CONDITION

As *r* depends on the ratio between Δt and Δx, the stability condition sets the maximum time step for a given space grid.

$$0 < \Delta t \leq \frac{1}{2} k (\Delta x)^2$$

NOTE: a larger value of *r* allows adopting a larger time step Δt.

When adopting r=1/2 (largest time step) it results:

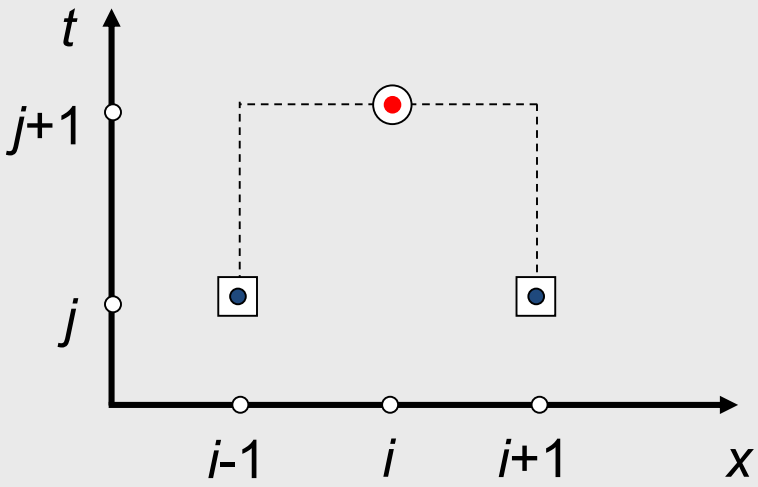
$$\Phi(i, j+1) = \frac{1}{2}\left[\Phi(i+1, j) + \Phi(i-1, j)\right]$$

The value of Φ at time step j+1 depends only on two values of Φ at the previous time step j.

COMPUTATIONAL SCHEME

value to determine

known values of Φ



ITERATIVE SOLUTION / 5

The stability issue can be solved by adopting the **implicit formula**, proposed by Crank and Nicholson in 1974, which is unconditionally stable, for any value of *r*.

In the implicit formula, the space derivative is replaced by the average of the central difference at time step j e $j+1$:

$$k\frac{\Phi(i,j+1)-\Phi(i,j)}{\Delta t}=\frac{1}{2}\left[\frac{\Phi(i+1,j)-2\Phi(i,j)+\Phi(i-1,j)}{(\Delta x)^2}+\right.$$
$$\left.+\frac{\Phi(i+1,j+1)-2\Phi(i,j+1)+\Phi(i-1,j+1)}{(\Delta x)^2}\right]$$

After re-ordering, we obtain:

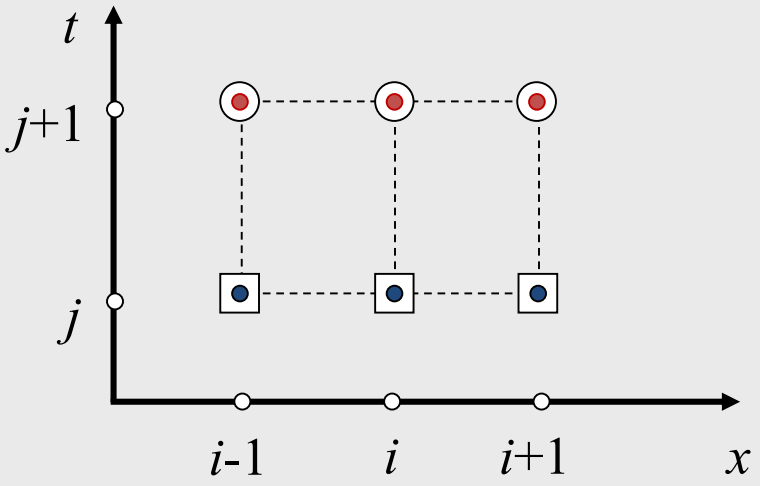
$$-r\,\Phi(i-1,j+1)+2(1+r)\,\Phi(i,j+1)-r\,\Phi(i+1,j+1)=$$
$$=r\,\Phi(i-1,j)+2(1-r)\,\Phi(i,j)+r\,\Phi(i+1,j)$$

Three values of Φ at time step j+1 depend on three values of Φ at time step j.

COMPUTATIONAL SCHEME

value to determine

known values of Φ



ITERATIVE SOLUTION / 7

The determination of the values of Φ through the implicit formula requires the set the equation in N nodes (i=1..N) and to solve a system of N equation with N unknowns.

The implicit formula is unconditionally stable and it allows to adopt larger time step Δt compared to the explicit formula.

A convenient choice is typically r=1, as it reduces the equation to the following:

$$-\Phi(i-1,j+1)+4\,\Phi(i,j+1)-\Phi(i+1,j+1)=$$
$$=\Phi(i-1,j)+\Phi(i+1,j)$$

ACCURACY

The three major sources of error in the numerical solution of physical problems are:

1. Modeling error (approximation in the transformation of the real world problem into a model)
2. Discretization error (e.g., due to the transformation of the partial derivatives into finite differences)
3. Rounding error (due to the finite precision of digital computers)

NOTE: Accuracy increases when the grid size becomes smaller, but there is a limit set by the rounding error.