Lecture 3

APPLICATION OF THE FDTD METHOD TO MAXWELL'S EQUATIONS

OUTLINE

1. Application of the FDTD method to the Maxwell's equations
2. Yee space-time map
3. Definition of the source
4. The boundary conditions

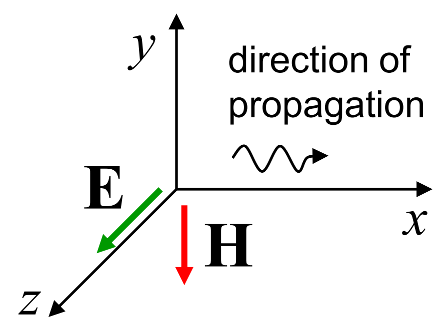
MAXWELL'S EQUATIONS (1D CASE)

Let us consider Maxwell's equation in a isotropic and linear medium, without any source:

$$\begin{cases} \nabla \times \mathbf{E} = -\mu \dfrac{\partial \mathbf{H}}{\partial t} & \text{(1ST EQUATION)} \\ \nabla \times \mathbf{H} = \varepsilon \dfrac{\partial \mathbf{E}}{\partial t} + \sigma \mathbf{E} & \text{(2ND EQUATION)} \end{cases}$$

In the 1D case, a solution consists in a TEM wave propagating in the x direction, with $\mathbf{E}=E_z\mathbf{z}$ and $\mathbf{H}=-H_y\mathbf{y}$. The Maxwell's equations result:

$$\begin{cases} \dfrac{\partial E_z}{\partial x} = -\mu \dfrac{\partial H_y}{\partial t} \\ -\dfrac{\partial H_y}{\partial x} = \varepsilon \dfrac{\partial E_z}{\partial t} + \sigma E_z \end{cases}$$



First Equation / 1

1st equation

After defining a grid with space step Δx and time step Δt, the derivatives in the 1st equation are approximated by the central difference

$$\frac{\partial H_y(i\Delta x, n\Delta t)}{\partial t} \cong \frac{H_y(i, n+1/2) - H_y(i, n-1/2)}{\Delta t}$$

$$\frac{\partial E_z(i\Delta x, n\Delta t)}{\partial x} \cong \frac{E_z(i+1/2, n) - E_z(i-1/2, n)}{\Delta x}$$

FIRST EQUATION / 2

By substituting in the 1st equation, it results:

$$\frac{H_y(i,n+1/2)-H_y(i,n-1/2)}{\Delta t}=-\frac{1}{\mu}\frac{E_z(i+1/2,n)-E_z(i-1/2,n)}{\Delta x}$$

After re-ordering with respect to the most recent time value:

$$H_y(i,n+1/2)=H_y(i,n-1/2)-\frac{\Delta t}{\mu\Delta x}\left[E_z(i+1/2,n)-E_z(i-1/2,n)\right]$$

It is observed that electric and magnetic fields are interlaced in space and time (i.e., samples are computed with space step $\Delta x/2$ and time step $\Delta t/2$).

FIRST EQUATION / 3

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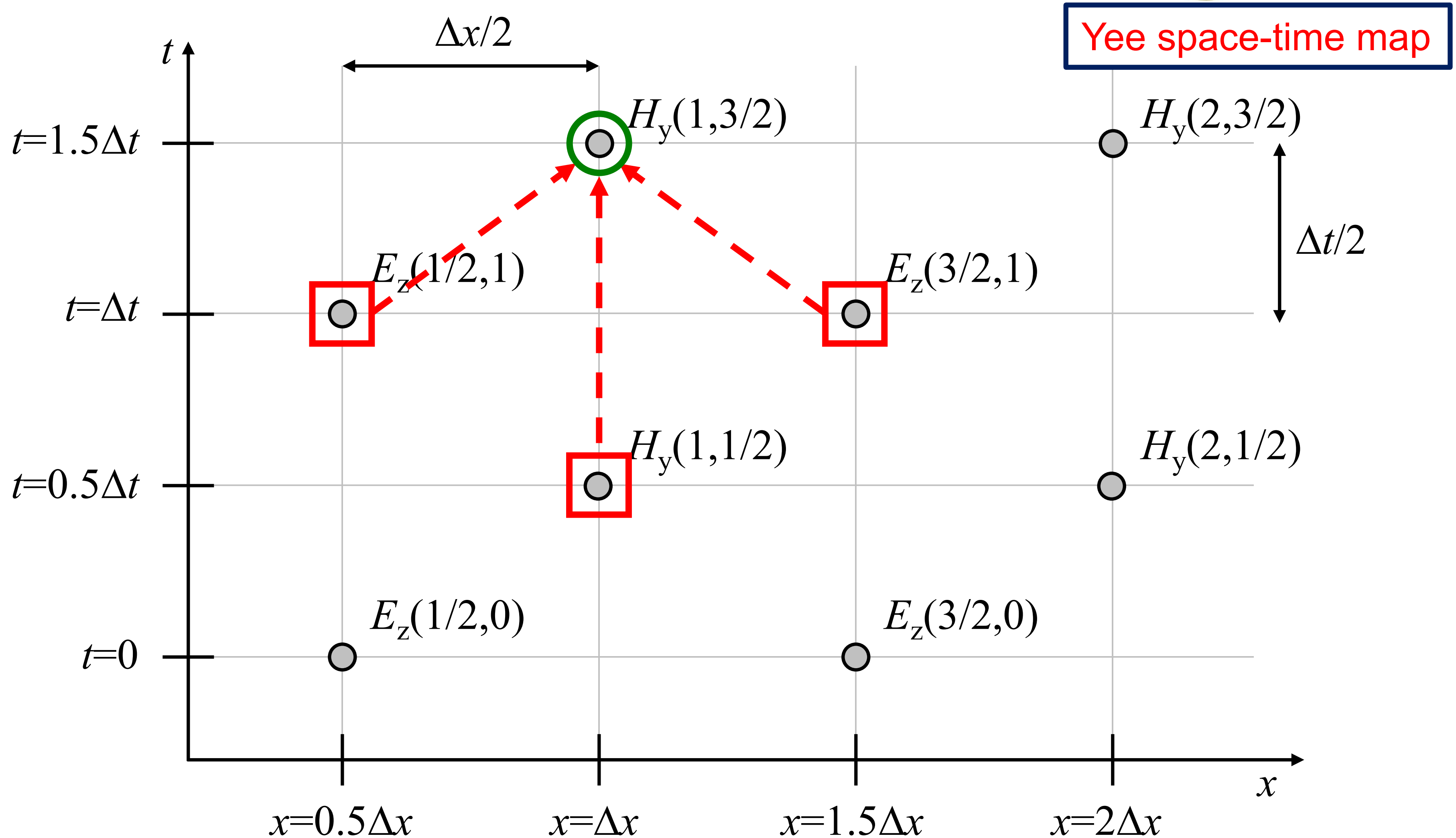


* K.S. Yee, "Numerical Solution of Initial Boundary Value Problems Involving Maxwell's Equations in Isotropic Media," *IEEE Trans Antennas Propagat.*, Vol. AP-14, No. 3, pp. 302–307, May 1966.

First Equation / 4

For implementation purposes, in order to optimize the computational efficiency, the equation can be recast as follows:

$$H_y(i,n+1/2) = H_y(i,n-1/2) - \widetilde{E}_z(i+1/2,n) + \widetilde{E}_z(i-1/2,n)$$

where

$$\widetilde{E}_z(.,.) = R_b E_z(.,.) \qquad R_b = \frac{\Delta t}{\mu \Delta x}$$

Parameter R_b only depends on the position (through the magnetic permeability μ), and it can be computed once for all points before the time iteration, with significant time saving.

If the material has no magnetic properties, R_b is constant value everywhere.

SECOND EQUATION / 1

2nd equation

The 1st equation allows computing $H_y(x,t+\Delta t/2)$ once the values $H_y\,(x,\ t-\Delta t/2)$ and $E_z(x\pm\Delta x/2,t)$ are known.
The calculation of $E_z(x\pm\Delta x/2,t)$ requires the solution of the second Maxwell's equation.

To obtain an iterative procedure, the second equation is used, and its derivatives are approximated by the central difference around al space point $i+1/2$ e and the time step $n+1/2$. It results:

$$-\frac{H_y(i+1,n+1/2)-H_y(i,n+1/2)}{\Delta x}=$$

$$=\varepsilon\frac{E_z(i+1/2,n+1)-E_z(i+1/2,n)}{\Delta t}+\sigma E_z(i+1/2,n+1/2)$$

SECOND EQUATION / 2

After reordering the equation, it results:

$$E_z(i+1/2,n+1)+\frac{\sigma\Delta t}{\varepsilon}E_z(i+1/2,n+1/2)-E_z(i+1/2,n)=$$

$$=-\frac{\Delta t}{\varepsilon\Delta x}\left[H_y(i+1,n+1/2)-H_y(i,n+1/2)\right]$$

NOTE: in this equation there are values calculated at three time steps (n+1, n+1/2, n).

To avoid additional time steps, the electric field $E_z(i+1/2,n+1/2)$ is obtained by linear interpolation:

$$E_z(i+1/2,n+1/2)=\frac{E_z(i+1/2,n+1)+E_z(i+1/2,n)}{2}$$

SECOND EQUATION / 3

After replacing the term in the equation, it results:

$$\left[1+\frac{\sigma\Delta t}{2\varepsilon}\right]E_z(i+1/2,n+1)-\left[1-\frac{\sigma\Delta t}{2\varepsilon}\right]E_z(i+1/2,n)=$$

$$=-\frac{\Delta t}{\varepsilon\Delta x}\left[H_y(i+1,n+1/2)-H_y(i,n+1/2)\right]$$

After re-ordering with respect to the most recent time value:

$$E_z(i+1/2,n+1)=$$

$$=\frac{\left[1-\frac{\sigma\Delta t}{2\varepsilon}\right]}{\left[1+\frac{\sigma\Delta t}{2\varepsilon}\right]}E_z(i+1/2,n)-\frac{\frac{\Delta t}{\varepsilon\Delta x}}{\left[1+\frac{\sigma\Delta t}{2\varepsilon}\right]}\left[H_y(i+1,n+1/2)-H_y(i,n+1/2)\right]$$

SECOND EQUATION / 4

Similarly to the case of the 1st equation, this equation can be recast as follows:

$$\tilde{E}_z(i+1/2,n+1) =$$
$$= C_a \tilde{E}_z(i+1/2,n) - C_b\left[H_y(i+1,n+1/2) - H_y(i,n+1/2)\right]$$

where

$$C_a = \frac{\left[1-\frac{\sigma\Delta t}{2\varepsilon}\right]}{\left[1+\frac{\sigma\Delta t}{2\varepsilon}\right]} \qquad C_b = \frac{R_b\frac{\Delta t}{\varepsilon\Delta x}}{\left[1+\frac{\sigma\Delta t}{2\varepsilon}\right]}$$

Parameters C_a and C_b depend on the position only (through ε, μ and σ), and they can be computed once for all points before the time iteration.

MAXWELL'S EQUATIONS (3D CASE)

The extension to the 3D case is only formally more complex (6 equations).

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right)$$

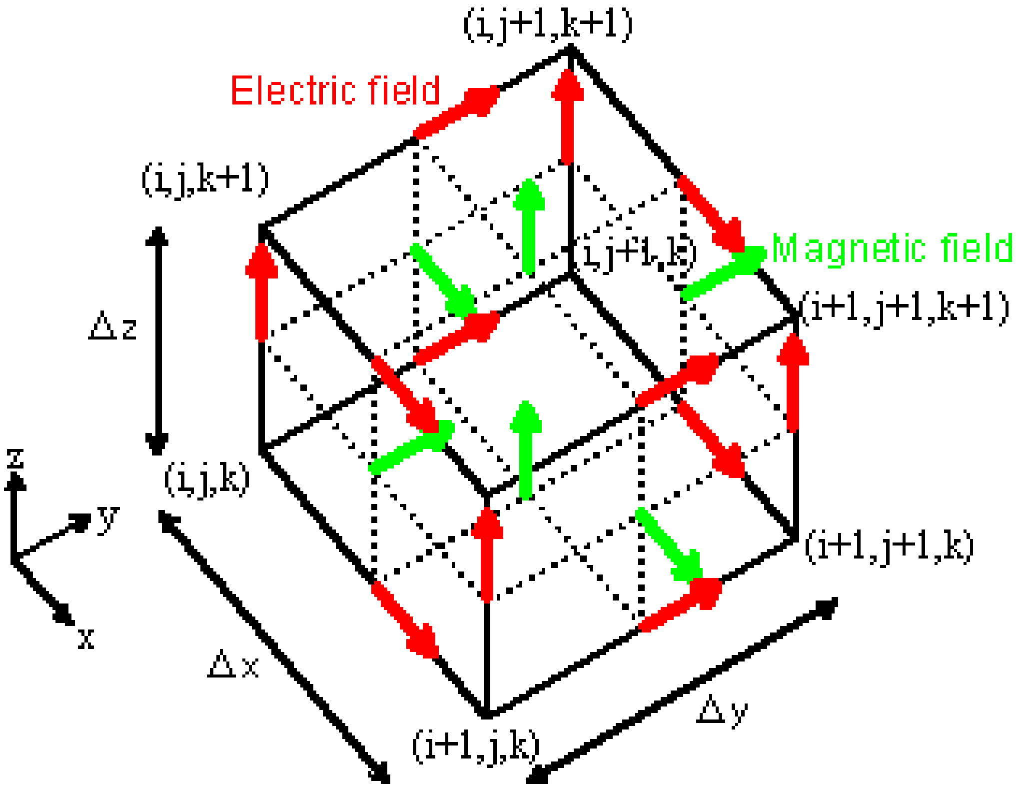
$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \sigma E_x\right)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} - \sigma E_y\right)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma E_z\right)$$



YEE CELL

FDTD Sources (1D Case) / 1

The initial conditions for the FDTD problem are obtained by defining the source field, which represents the excitation of the system.

The source field can be defined in two ways:

A. Transient pulse
(suitable for broadband analysis)

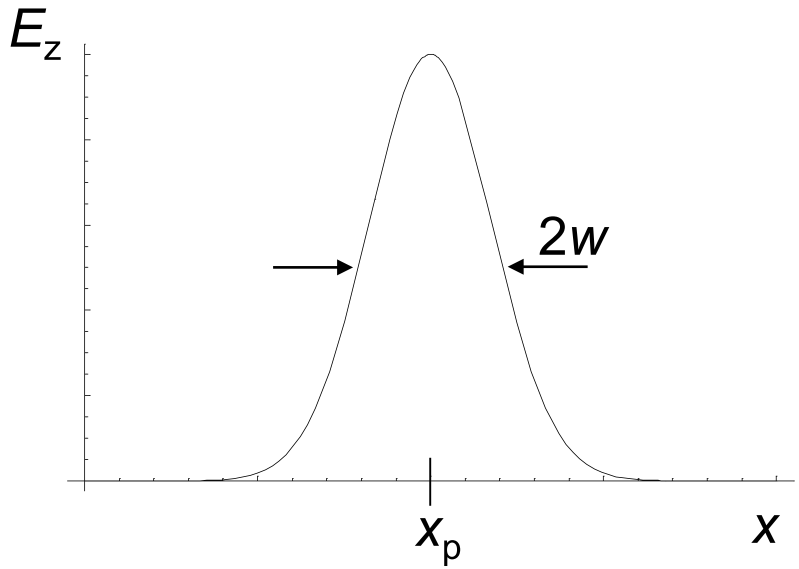
B. Sinusoidal source
(suitable for single-frequency analysis)

A. Transient pulse

The incident wave can be a transient pulse, e.g. with a gaussian shape.

$$E_z(x,0) = E_0 \exp\left[-\frac{(x-x_p)^2}{w^2}\right]$$

$$H_y(x,0) = \frac{E_0}{Z_0} \exp\left[-\frac{(x-x_p+\Delta x/4)^2}{w^2}\right]$$



This type of source is defined in the entire computation domain as an initial condition. The source is not updated, and it propagates through the computational medium according to Maxwell's equations.

FDTD SOURCES (1D CASE) / 3

B. Sinusoidal source

The electric field in a given location x_i has a sinusoidal time variation

$$E_z(x_i, t) = E_0 \sin(2\pi f_0 t)$$

with frequency f_0.

This source be thought of as a radiating dipole in the sense that the field oscillates sinusoidally at that point for the duration of time marching.

In this case, the source is updated during the time marching.

ABSORBING BOUNDARY / 1

In open/radiation problems, we must apply an absorbing boundary at the truncation of the computational region, which should minimize reflections.

A. Absorbing Boundary Condition (ABC)

An absorbing boundary condition can be obtained by discretizing the wave equation at the endpoints of the region (Mur boundary condition)

$$\frac{\partial^2 E_z}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0$$

which can be expressed in the form:

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right)E_z(x,t) = 0$$

The solution is given by two waves, propagating in the positive and negative directions of the x axis when time increases, respectively.

ABSORBING BOUNDARY / 2

The following equation is applied in the right endpoint (N-th node):

$$\left(\frac{\partial}{\partial x}+\frac{1}{c}\frac{\partial}{\partial t}\right)E_z(x,t)=0$$

By discretizing the equation in the N-1/2 space point and at n time step:

$$\frac{E_z(N,n)-E_z(N-1,n)}{\Delta x}+\frac{1}{c}\frac{E_z(N-1/2,n+1/2)-E_z(N-1/2,n-1/2)}{\Delta t}=0$$

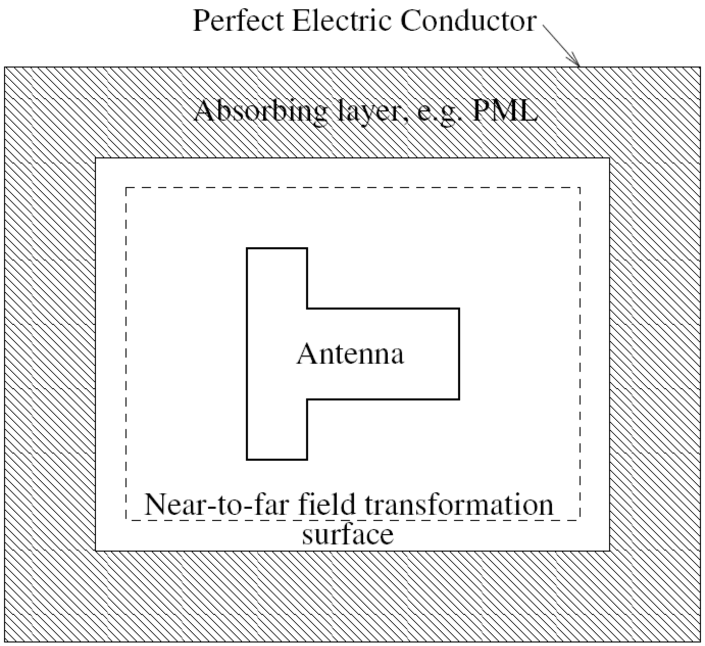
After interpolating the samples calculated at 1/2 time step or space point, it results:

$$E_z(N,n)=E_z(N-1,n-1)+\left(\frac{c\Delta t-\Delta x}{c\Delta t+\Delta x}\right)\left[E_z(N-1,n)-E_z(N,n-1)\right]$$

By using the magic time step $\Delta t=\Delta x/c$, it finally results:

$$\boxed{E_z(N,n)=E_z(N-1,n-1)}$$

B. Perfectly Matched Layer (PML)



Proposed for the first time by Bérenger in 1994, the PML method consists in adding a thin layer of lossy anisotropic, non-physical material at the boundary of the computational domain.

This method provides better performance than the ABCs, especially for oblique incidence, but it (slightly) increases the dimension of the computational domain.

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* J.-P. Berenger, "A perfectly matched layer for the absorption of electromagnetic waves," *J. Comput. Phys*., Vol. 114, No. 1, pp. 185-200, 1994.