Lecture 4

NUMERICAL DISPERSION AND STABILITY OF THE FDTD METHOD

OUTLINE

1. Numerical dispersion
2. Stability of the FDTD method

NUMERICAL DISPERSION AND STABILITY

The choice of space step Δx and time step Δt determine the accuracy and stability of the FDTD.

Besides the topics already discussed (description of the geometry and of the field variation), two other issues need to be carefully accounted for:

1. numerical dispersion: under certain conditions, it may happen that waves at different frequency propagate with different phase velocity (even in vacuum);

2. stability: the space step Δx and time step Δt need to be properly chosen, to avoid that the solution becomes arbitrarily large after a certain number of time steps (due to the increasing error generated by the FDTD approximation).

NUMERICAL DISPERSION / 1

Dispersion is the relation between the wave number and the frequency.

To examine the numerical dispersion in the FDTD method, let us consider the wave equation in one dimension:

$$\frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2} = 0$$

whose solution is:

$$U(x,t) = U_0 \, e^{j(\omega t - kx)}$$

where:

ω	is the angular frequency,
$k = \omega/c$	is the wave number (linear relation between k and ω),
$v_p = \omega/k = c$	is the phase velocity (in vacuum).

Numerical Dispersion / 2

The investigation of numerical dispersion is performed through the FDTD solution of the wave equation.

A grid of points with space step Δx and time step Δt is defined.

The time and space derivative in the wave equation are approximated with the central difference.

$$\frac{\partial^2 U}{\partial t^2}(i\Delta x, n\Delta t) \cong \frac{U(i,n+1)-2U(i,n)+U(i,n-1)}{\Delta t^2}$$

$$\frac{\partial^2 U}{\partial x^2}(i\Delta x, n\Delta t) \cong \frac{U(i+1,n)-2U(i,n)+U(i-1,n)}{\Delta x^2}$$

NUMERICAL DISPERSION / 3

By replacing the central difference expressions in the wave equation, it results:

$$\frac{U(i,n+1)-2U(i,n)+U(i,n-1)}{\Delta t^2}-c^2\frac{U(i+1,n)-2U(i,n)+U(i-1,n)}{\Delta x^2}=0$$

After re-ordering with respect to the most recent time value:

$$U(i,n+1)=$$

$$=\left(\frac{c\Delta t}{\Delta x}\right)^2\left[U(i+1,n)-2U(i,n)+U(i-1,n)\right]+2U(i,n)-U(i,n-1)$$

NUMERICAL DISPERSION / 4

Subsequently, the a-priori information on the solution of the wave equation is exploited:

$$U(i,n) = U_0\, e^{j(\omega n \Delta t - \tilde{k} i \Delta x)}$$

where $\tilde{k}$ represents the wavenumber at the angular frequency ω, obtained through the FDTD solution of the wave equation.

This solution is incorporated in the discretized equation. After some algebraic manipulation, it finally results that the dispersion relation between k and ω is

$$\cos(\omega \Delta t) = \left(\frac{c \Delta t}{\Delta x} \right)^2 \left[\cos(\tilde{k} \Delta x) - 1 \right] + 1$$

NUMERICAL DISPERSION / 5

$$\cos(\omega\Delta t) = \left(\frac{c\Delta t}{\Delta x}\right)^2 \left[\cos(\tilde{k}\Delta x) - 1\right] + 1$$

It is noted that this relation is non-linear and depends on the choice of the space step Δx and the time step Δt.

Since this relation is non-linear, the numerical solution is affected by **phase errors**, because the phase velocity changes with frequency

CASE 1 – In the case $\Delta t \to 0$ and $\Delta x \to 0$, the cosine is approximated by its Taylor series, this resulting

$$1-\frac{(\omega\Delta t)^2}{2}=\left(\frac{c\Delta t}{\Delta x}\right)^2\left[1-\frac{(\tilde{k}\Delta x)^2}{2}-1\right]+1 \quad \Rightarrow \quad \boxed{\tilde{k}=\pm\frac{\omega}{c}}$$

CASE 2 – In the case $c\Delta t = \Delta x$ (magic time step), it results

$$\cos(\omega\Delta t)=\left(\frac{c\Delta t}{\Delta x}\right)^2\left[\cos(\tilde{k}\Delta x)-1\right]+1 \quad \Rightarrow \quad \boxed{\tilde{k}=\pm\frac{\omega}{c}}$$

CASE 3 – In the most general case, the dispersion relation is non-linear.

$$\cos(\omega\Delta t)=\left(\frac{c\Delta t}{\Delta x}\right)^2\left[\cos(\tilde{k}\Delta x)-1\right]+1$$

NUMERICAL DISPERSION / 7

University of Pavia

If the dispersion relation is non-linear, the phase velocity depends on frequency:

$$\tilde{v}_p = \frac{\omega}{\tilde{k}} = \frac{2\pi c}{\cos^{-1}\left\{1+\left(\frac{\Delta x}{c\Delta t}\right)^2\left[\cos(\omega\Delta t)-1\right]\right\}}\frac{\Delta x}{\lambda_0}$$

The phase velocity depends on the time and space step. For instance, if $\Delta x=\lambda_0/10$ e $c\Delta t=\Delta x/2$, it results:

$$\tilde{v}_p = \frac{\omega}{\tilde{k}} = 0.9873c$$

In the propagation over a distance of 10 λ_0, the phase error due to numerical dispersion can be significantly large:

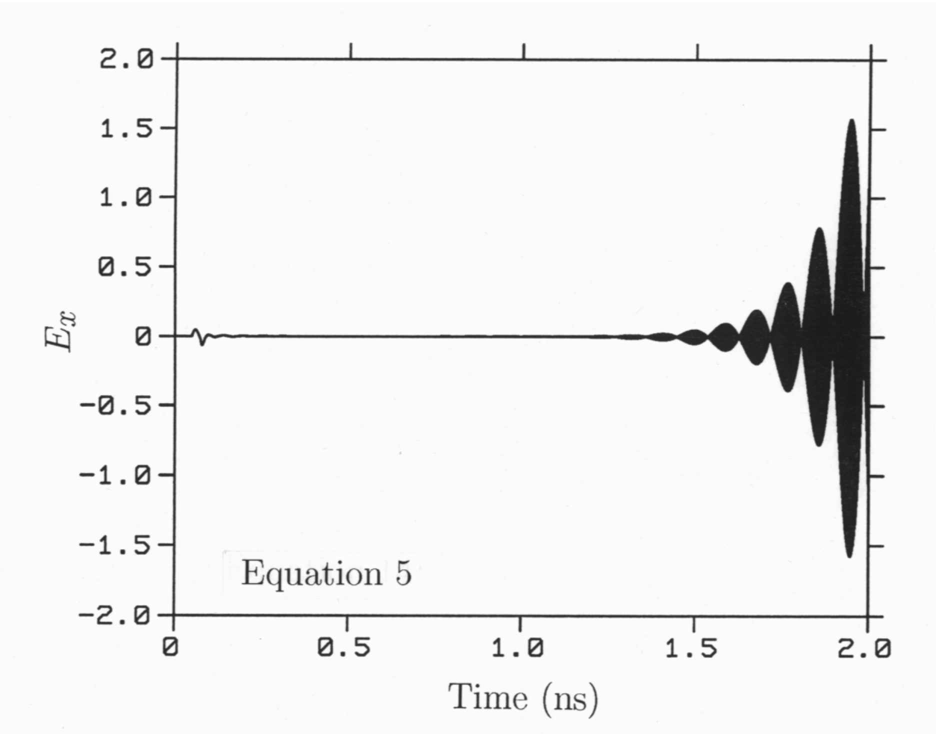
- if $\Delta x=\lambda_0/10$ and $c\Delta t=\Delta x/2$: $\tilde{v}_p = 0.9873\,c$ phase error=45.72°
- if $\Delta x=\lambda_0/20$ and $c\Delta t=\Delta x/2$: $\tilde{v}_p = 0.9968\,c$ phase error=11.19°

INSTABILITY

In order to guarantee numerical stability for the FDTD method, the upper limit of the time step Δt must be bounded by a criterion which restricts an update cycle's fields propagation from cell to cell being faster than allowed by the phase velocity in the medium.

A stability analysis for FDTD was performed, and provided the Courant-Friedrichs-Lewy criterion

$$c\Delta t \leq \frac{1}{\sqrt{\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}+\frac{1}{\Delta z^2}}}$$



STABILITY IN FDTD METHOD / 1

Time step Δt

As discussed in the previous section, the solution of the wave equation can be expressed in the form:

$$U(x,t) = U_0(x)\, e^{j\omega t}$$

where only the time dependency is explicitely shown.

This solution satisfies the following differential equation:

$$\frac{\partial U}{\partial t} = j\omega U$$

Stability in FDTD Method / 2

After approximating the equation by the central difference, it results:

$$\frac{U(n+1/2)-U(n-1/2)}{\Delta t} = j\omega U(n)$$

When Δt is small, we can define a numerical increment factor q

$$q = \frac{U(n+1/2)}{U(n)} = \frac{U(n)}{U(n-1/2)}$$

Stability is guaranteed under the condition $|q| \le 1$.

Introducing q in the discretized differential equation and reordering, the following quadratic equation is obtained:

$$q^2 - j\omega\Delta t\, q - 1 = 0$$

Stability in FDTD Method / 3

The solution to this equation is

$$q = \frac{j\omega\Delta t}{2} \pm \sqrt{1 - \left(\frac{\omega\Delta t}{2}\right)^2}$$

Numerical stability requires that $|q| \leq 1$. A sufficient condition is the following:

$$\frac{\omega\Delta t}{2} < 1 \quad \Rightarrow \quad \Delta t < \frac{2}{\omega} \quad \Rightarrow \quad \boxed{\Delta t < \frac{T}{\pi}}$$

where $T=2\pi/\omega$ represents the period of the wave.

STABILITY IN FDTD METHOD / 4

Space step Δx

The homogenous scalar wave equation in one-dimensional case for harmonic waves can be expressed as:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\omega^2}{c^2} U = 0$$

and the solution to this equation is a plane wave of the form

$$U(x,t) = U_0 \, e^{j(\omega t - kx)}$$

The second derivative can be approximated by using the *central difference* method:

$$\frac{\partial^2 U}{\partial x^2} \cong \frac{U(x+\Delta x) - 2U(x) + U(x-\Delta x)}{(\Delta x)^2}$$

Stability in FDTD Method / 5

Substituting this expression into a discretized equation, it results:

$$\frac{\partial^2 U}{\partial x^2} \cong \frac{e^{jk\Delta x} - 2 + e^{-jk\Delta x}}{(\Delta x)^2} U(x) = -\frac{\sin^2(k\Delta x/2)}{(\Delta x/2)^2} U(x)$$

By introducing the obtained expression in the initial differential equation:

$$\frac{\sin^2(k\Delta x/2)}{(\Delta x/2)^2} - \frac{\omega^2}{c^2} = 0$$

Multiplying by $(c\Delta t/2)^2$ and exploiting the relation $\omega\Delta t/2 < 1$ previously achieved, it results:

$$\left(\frac{c\Delta t}{2}\right)^2 \frac{\sin^2(k\Delta x/2)}{(\Delta x/2)^2} = \left(\frac{\omega\Delta t}{2}\right)^2 \leq 1$$

Stability in FDTD Method / 6

As the sin function is bounded by 1, the following condition must be met, in order for the above relation to be satisfied for any value of *k*.

$$\left(\frac{c\Delta t}{\Delta x}\right)^2 \leq 1$$

which reduces to the stability condition:

$$c\Delta t \leq \Delta x$$

NOTE: For higher dimensions, the stability condition becomes:

$$c\Delta t \leq \frac{\Delta x}{\sqrt{D}}$$

where D=1,2,3 is the dimension of the considered space.