Lecture 5

THE FDTD METHOD: EXAMPLES AND APPLICATIONS

INTRODUCTION

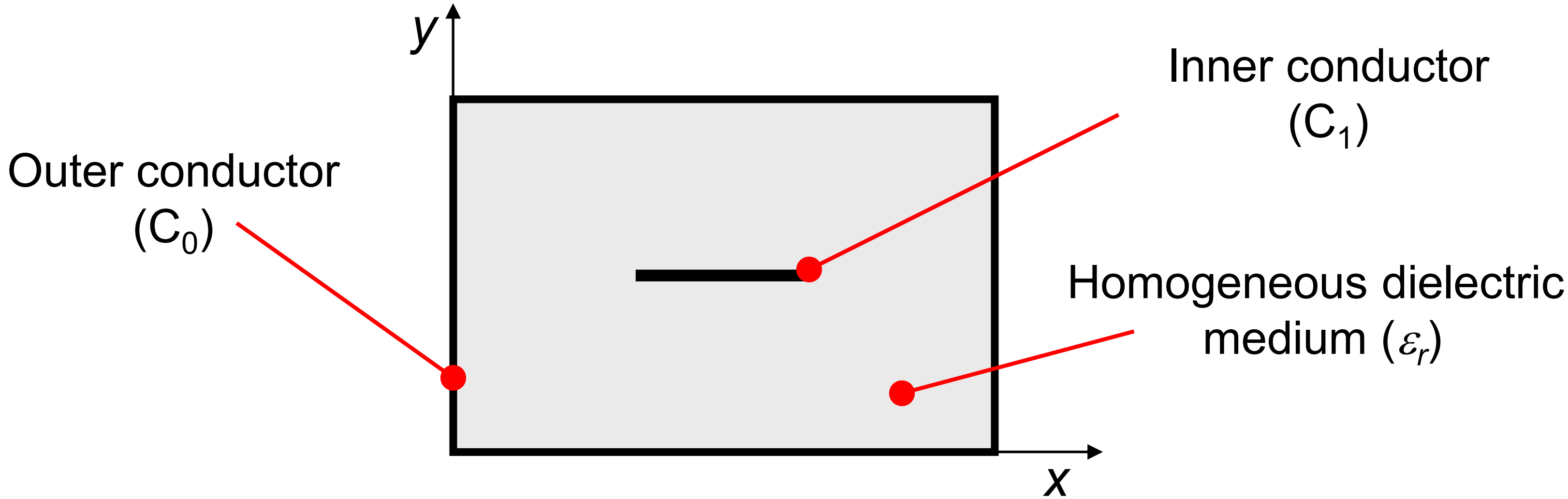
In this lecture, the FDTD method is applied to three cases of practical interest:

1. Calculation of the TEM mode characteristics of a shielded stripline
2. Modeling of a transmission line
3. Calculation of the modes of a metallic waveguide

The achieved results represent the starting point for the numeric implementation of computer codes.

TEM Mode of a Shielded Stripline/1

The FDTD method is applied to the calculation of the TEM mode characteristics of a shielded stripline.



The TEM mode of this structure can be determined through the solution of the Laplace equation, with the proper boundary condition:

$$\nabla_T^2 V = 0 \quad \text{in } S$$

$$\begin{cases} V = 0 & \text{in } C_0 \\ V = 1 & \text{in } C_1 \end{cases}$$

TEM MODE OF A SHIELDED STRIPLINE/2

In a Cartesian system, Laplace equation can be expressed in the form:

$$\nabla_T^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

By applying the *central difference* method, and using a mesh grid with size Δx e Δy, in direction *x* and *y*, respectively, it results:

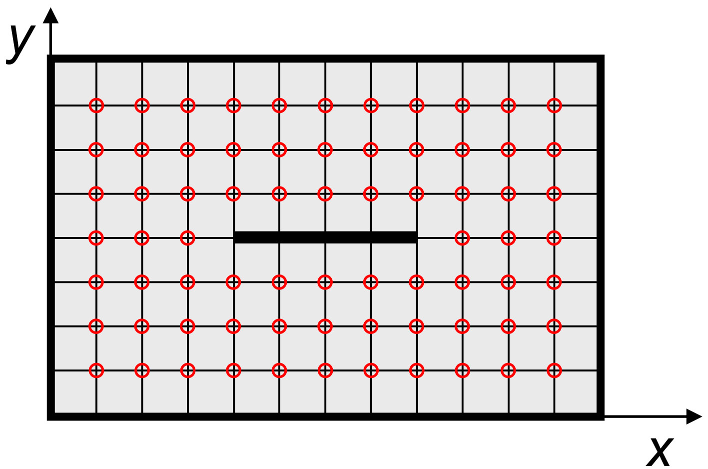
$$\frac{V(i+1,j) - 2V(i,j) + V(i-1,j)}{\Delta x^2} + \\ + \frac{V(i,j+1) - 2V(i,j) + V(i,j-1)}{\Delta y^2} = 0$$

TEM Mode of a Shielded Stripline/3

If $\Delta x = \Delta y$, it results:

$$V(i,j) = \frac{1}{4}\left[V(i+1,j) + V(i-1,j) + V(i,j+1) + V(i,j-1)\right]$$

This equation is applied to all inner grid nodes, thus leading to a system of equations.



The value of function V in the boundary nodes (conductors C_0 e C_1) is determined by the boundary conditions.

The solution of the system of equations provides the values of V in all grid nodes.

TEM Mode of a Shielded Stripline/4

Based on the obtained results, the characteristic impedance of the TEM mode can be computed.

The characteristic impedance Z_0 is defined as:

$$Z_0 = \sqrt{\frac{L}{C}}$$

where *L* and *C* represent the inductance and capacitance per unit length, respectively. The phase velocity *v* is given by:

$$v = \frac{1}{\sqrt{LC}}$$

By expressing *L* in terms of *v* and replacing it in the formula of Z_0, it results:

$$Z_0 = \frac{1}{Cv}$$

TEM Mode of a Shielded Stripline/5

As the dielectric medium is homogeneous, *v* results:

$$v = \frac{c}{\sqrt{\varepsilon_r}}$$

Capacitance *C* can be computed by using the following relation:

$$C = \frac{Q}{V_d}$$

where *Q* represents the charge per unit length and V_d is the voltage between inner and outer conductors.

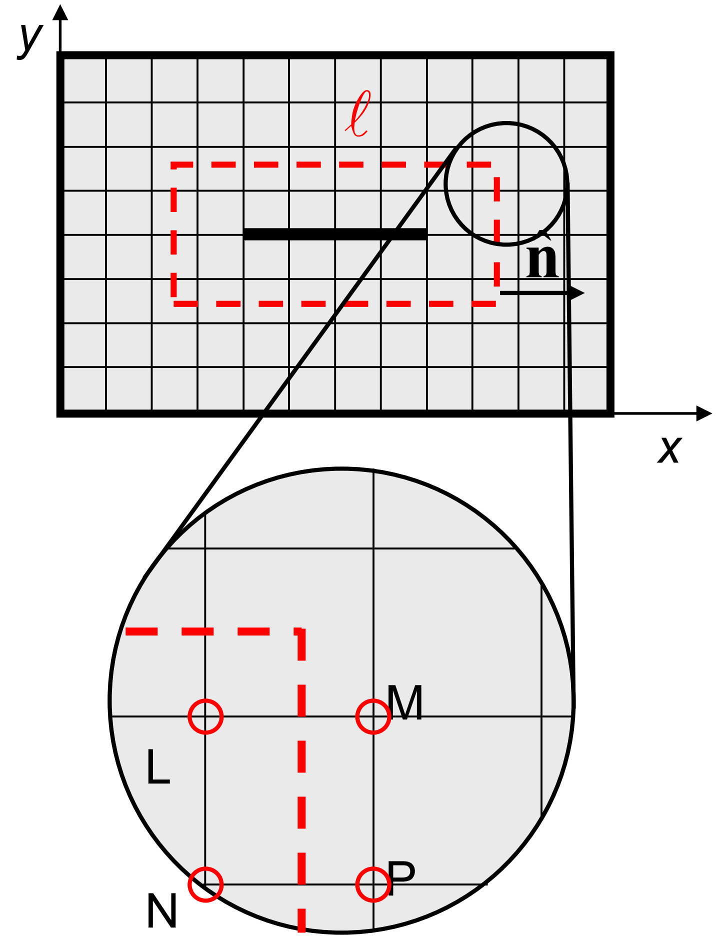
From the boundary condition it results:

$$V_d = 1$$

Charge Q is computed by appying the Gauss law along a closed path around the inner conductor:

$$Q = \oint_\ell \mathbf{D} \cdot \hat{\mathbf{n}}\, d\ell = \oint_\ell \varepsilon\, \mathbf{E} \cdot \hat{\mathbf{n}}\, d\ell =$$

$$= -\oint_\ell \varepsilon\, \nabla V \cdot \hat{\mathbf{n}}\, d\ell = -\oint_\ell \varepsilon \frac{\partial V}{\partial n} d\ell$$



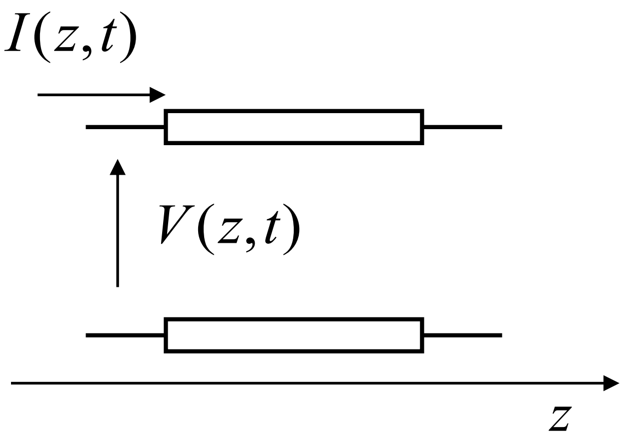
By discretizing the integral, it results:

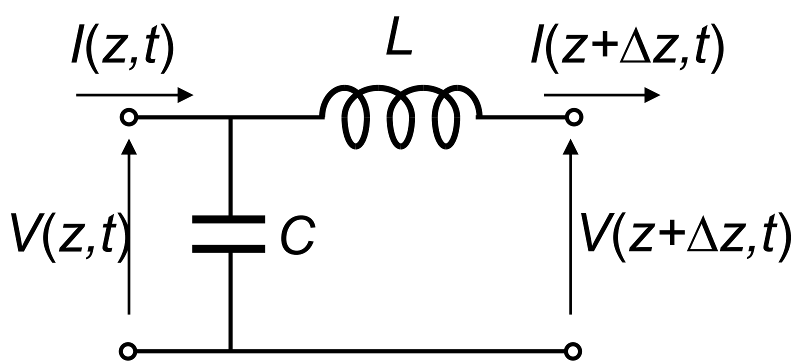
$$Q = -\varepsilon \left(\frac{V_P - V_N}{\Delta x} \Delta y + \frac{V_M - V_L}{\Delta x} \Delta y + \dots \right)$$

If $\Delta x = \Delta y$, it results:

$$Q = \varepsilon \left(V_N + V_L + \dots - V_P - V_M - \dots \right)$$

The FDTD method can be applied to the modeling of transmission lines.





$$V(z,t) - V(z+\Delta z,t) = L\frac{\partial I(z,t)}{\partial t}\Delta z$$

$$I(z,t) - I(z+\Delta z,t) = C\frac{\partial V(z,t)}{\partial t}\Delta z$$

$$\frac{\partial V(z,t)}{\partial z} = -L\frac{\partial I(z,t)}{\partial t}$$

$$\frac{\partial I(z,t)}{\partial z} = -C\frac{\partial V(z,t)}{\partial t}$$

MODELING OF A TRANSMISSION LINE / 2

The differential equations are discretized by using the *central difference* method, with space grid size Δz and time step Δt.

From the **first equation** it results:

$$\left.\frac{\partial V(z,t)}{\partial z}\right|_{z=(i-1/2)\Delta z,\, t=n\Delta t} = \frac{V(i,n)-V(i-1,n)}{\Delta z}$$

$$\left.\frac{\partial I(z,t)}{\partial t}\right|_{z=(i-1/2)\Delta z,\, t=n\Delta t} = \frac{I(i-1/2,n+1/2)-I(i-1/2,n-1/2)}{\Delta t}$$

By replacing in the first equation, it results:

$$\frac{V(i,n)-V(i-1,n)}{\Delta z} = -L\,\frac{I(i-1/2,n+1/2)-I(i-1/2,n-1/2)}{\Delta t}$$

After re-ordering with respect to the most recent time value:

$$I(i-1/2,n+1/2) = I(i-1/2,n-1/2) - \frac{\Delta t}{L}\,\frac{V(i,n)-V(i-1,n)}{\Delta z}$$

Similarly, from the **second equation** it results:

$$\left.\frac{\partial I(z,t)}{\partial z}\right|_{z=(i-1)\Delta z,\, t=(n+1/2)\Delta t} = \frac{I(i-1/2,n+1/2)-I(i-3/2,n+1/2)}{\Delta z}$$

$$\left.\frac{\partial V(z,t)}{\partial t}\right|_{z=(i-1)\Delta z,\, t=(n+1/2)\Delta t} = \frac{V(i-1,n+1)-V(i-1,n)}{\Delta t}$$

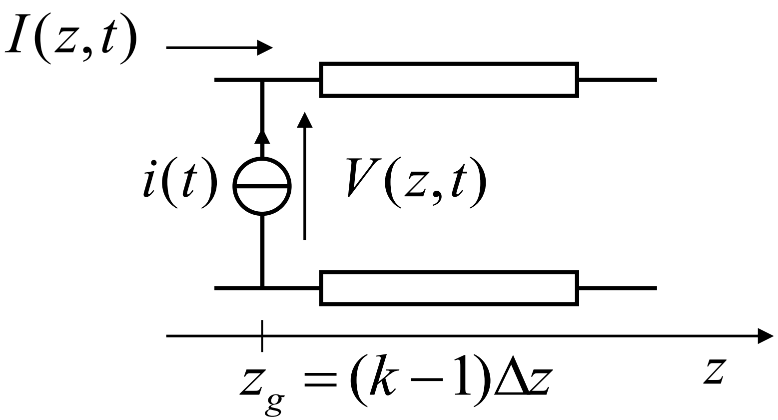
By replacing in the second equation, it results:

$$\frac{I(i-1/2,n+1/2)-I(i-3/2,n+1/2)}{\Delta z} = -C\frac{V(i-1,n+1)-V(i-1,n)}{\Delta t}$$

After re-ordering with respect to the most recent time value:

$$V(i-1,n+1) = V(i-1,n) - \frac{\Delta t}{C}\frac{I(i-1/2,n+1/2)-I(i-3/2,n+1/2)}{\Delta z}$$

If there is a current source at section z_g, the equations are modified in the following way:



$$\frac{\partial V(z,t)}{\partial z} = -L\frac{\partial I(z,t)}{\partial t}$$

$$\frac{\partial I(z,t)}{\partial z} = -C\frac{\partial V(z,t)}{\partial t} + \delta(z - z_g)\, i(t)$$

The discretized equations become:

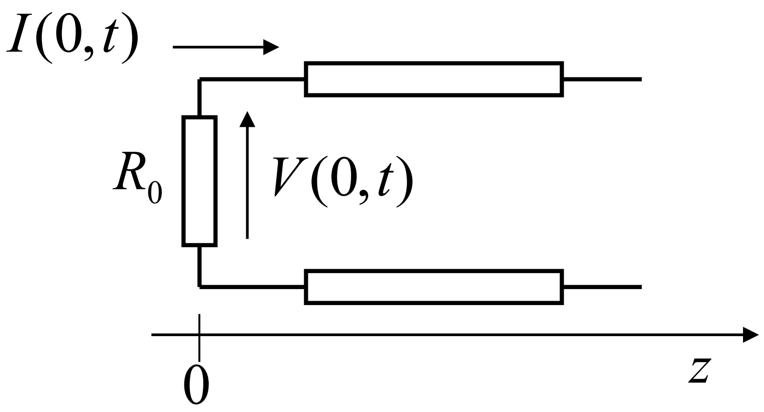
$$I(k-1/2, n+1/2) = I(k-1/2, n-1/2) - \frac{\Delta t}{L}\frac{V(k,n) - V(k-1,n)}{\Delta z}$$

$$V(k-1, n+1) = V(k-1, n) +$$
$$-\frac{\Delta t}{C}\frac{I(k-1/2, n+1/2) - I(k-3/2, n+1/2)}{\Delta z} + \frac{\Delta t}{C} i(n+1/2)$$

The boundary conditions are represented by the load resistances, at sections z=0 and z=L.

A. Case z=0

If we consider the load impedance R_0 at section z=0, it results:



$$V(0,t) = -R_0 I(0,t)$$

The second equation, discretized at z=0, needs to be modified.

The space derivative of the current in section *z*=0 is:

$$\left.\frac{\partial I(z,t)}{\partial z}\right|_{z=0,\, t=(n+1/2)\Delta t} = \frac{I(1/2, n+1/2) - I(0, n+1/2)}{\Delta z/2}$$

By exploiting the boundary condition (Ohm law), it results:

$$\left.\frac{\partial I(z,t)}{\partial z}\right|_{z=0,\, t=(n+1/2)\Delta t} = \frac{I(1/2, n+1/2) + V(0, n+1/2)/R_0}{\Delta z/2}$$

As the voltage is not computed at time step $(n+1/2)\Delta t$, the interpolated value is used:

$$\left.\frac{\partial I(z,t)}{\partial z}\right|_{z=0,\, t=(n+1/2)\Delta t} = \frac{I(1/2, n+1/2) + \left[V(0, n) + V(0, n+1)\right]/2R_0}{\Delta z/2}$$

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The time derivative of voltage is:

$$\left.\frac{\partial V(z,t)}{\partial t}\right|_{z=0,\,t=(n+1/2)\Delta t} = \frac{V(0,n+1)-V(0,n)}{\Delta t}$$

By replacing in the second equation, it results:

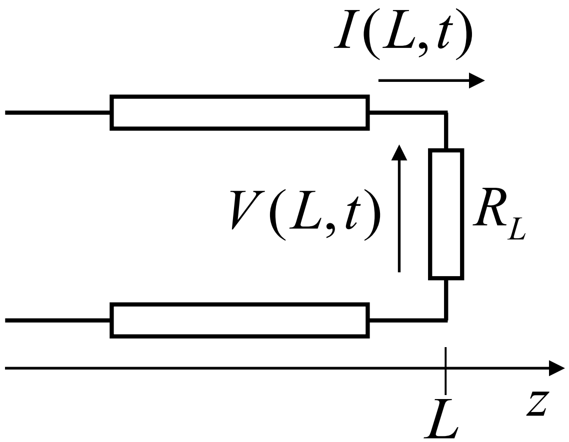
$$\frac{2I(1/2,n+1/2)+\left[V(0,n)+V(0,n+1)\right]/R_0}{\Delta z} = -C\frac{V(0,n+1)-V(0,n)}{\Delta t}$$

After re-ordering with respect to the most recent time value:

$$V(0,n+1) = \frac{R_0 C\Delta z - \Delta t}{R_0 C\Delta z + \Delta t}V(0,n) - \frac{2R_0\Delta t}{R_0 C\Delta z + \Delta t}I(1/2,n+1/2)$$

B. Case $z=L$

If we consider the load impedance R_L at section $z=L$, it results:



$$V(L,t) = R_L I(L,t)$$

Also in this case, the second equation, discretized at $z=L$, needs to be modified. It results:

$$V(N_z, n+1) = \frac{R_L C\Delta z - \Delta t}{R_L C\Delta z + \Delta t} V(N_z, n) + \frac{2R_L \Delta t}{R_L C\Delta z + \Delta t} I(N_z - 1/2, n+1/2)$$

Modes of a Metallic Waveguide / 1

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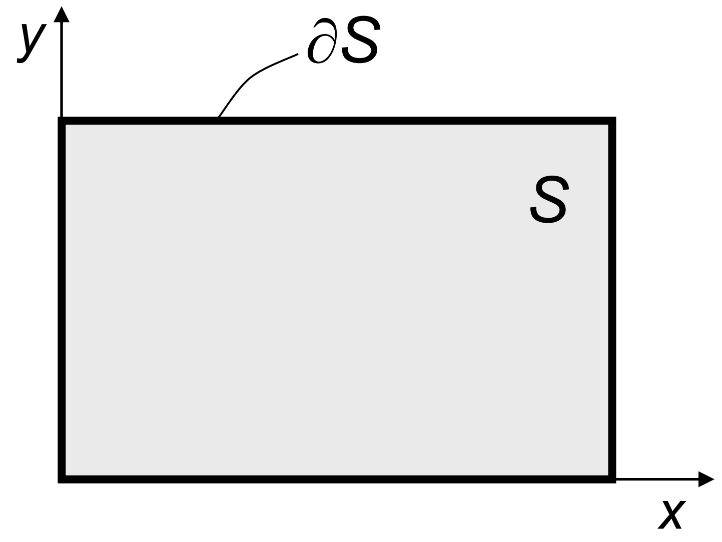
The calculation of the modes of a metallic waveguide can be implemented by using a finite difference method.

The waveguide modes can be expressed in terms of scalar potentials, which are the eigen-solutions of the Helmholtz equation with proper boundary conditions:

$$\begin{cases} \nabla_T^2 \Phi - k^2 \Phi = 0 & \text{in } S \\ \Phi = 0 & \text{su } \partial S \end{cases} \quad \text{(TM modes)}$$

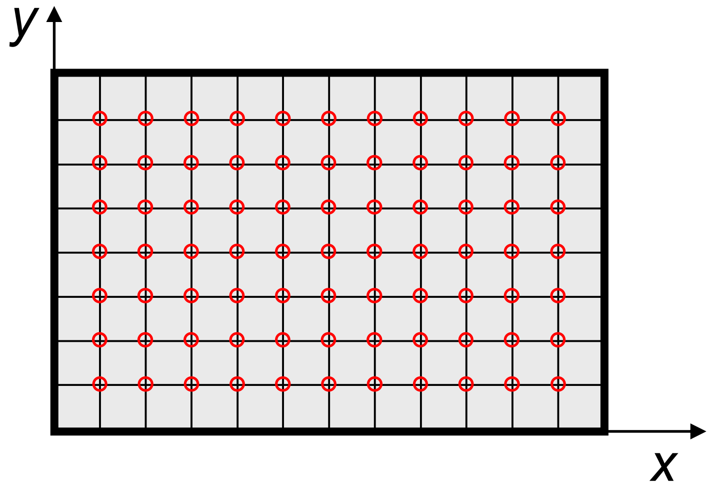
$$\begin{cases} \nabla_T^2 \Phi - k^2 \Phi = 0 & \text{in } S \\ d\Phi / dn = 0 & \text{su } \partial S \end{cases} \quad \text{(TE modes)}$$



The Helmholtz equation is discretized by adopting the *central difference* method. By using a space grid size $\Delta x=\Delta y=h$, it results:

$$\Phi(i+1,j)+\Phi(i-1,j)+\Phi(i,j+1)+\Phi(i,j-1)-(4-h^2k^2)\Phi(i,j)=0$$

NOTE: this equation is applied to all inner nodes.



For nodes located on the boundary, TM and TE cases require different treatments.

Modes of a Metallic Waveguide / 3

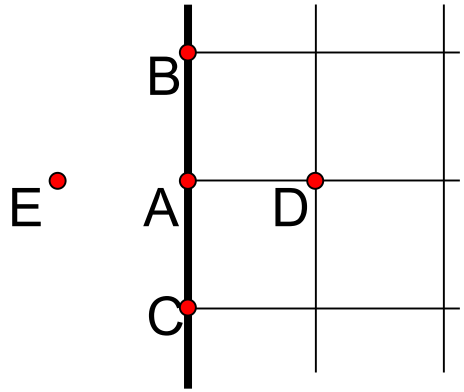
A. TM modes

In the case of TM modes, the Dirichlet condition is applied (Φ=0 on ∂S).

B. Modi TE

In the case of TE modes, the Neumann condition is applied ($d\Phi/dn$=0 on ∂S). In the case shown in the figure, it results:

$$\frac{\partial \Phi}{\partial n} = 0 \quad \Longrightarrow \quad \Phi_D = \Phi_E$$



It finally leads to:

$$\Phi_B + \Phi_C + 2\Phi_D - (4 - h^2k^2)\Phi_A = 0$$

Modes of a Metallic Waveguide / 4

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By applying the discretized Helmholtz equation and the boundary conditions to all grid nodes, a system of N equations is obtained, in the form:

$$(\mathrm{A} - \lambda \mathrm{I})\Phi = 0 \qquad \text{or} \qquad \mathrm{A}\Phi = \lambda\Phi$$

where A is an $N \times N$ matrix and I is the identity matrix with size $N \times N$.

The solution of the systems yields:

eigen-vectors Φ (related to the values of modal voltage in the grid nodes)

eigen-values λ (related to the modal cutoff wave-numbers).

NOTE: The solution of this problem can be based on a direct method: by imposing $|\mathrm{A}-\lambda \mathrm{I}|=0$, a polynomial function in λ is obtained, whose zeros provide the eigen-values of the problem.