Lecture 6

THE METHOD OF MOMENTS (MoM)

INTRODUCTION

The Method of Moments (MoM) allows trasforming an integro-differential functional equation into an algebraic matrix equation, which can be solved by using a digital computer.

Applied for the first time to electromagnetic problems in 1967, nowadays the MoM is adopted in several numerical techniques for the solution of electromagnetic problems.

The applicability of the method extends well beyond the field of em problems, e.g. to mechanics, fluido-dynamics, ….

Useful readings on the Method of Moments:

R. F. Harrington, "Matrix Methods for Field Problems," *Proc. IEEE*, Feb. 1967.
R. F. Harrington, *Field Computation by Moment Methods*, IEEE Press, 1993.

INTRODUCTION

In the application to em problems, the MoM aims at the determination of fields or currents related to the solutions of the Maxwell's equations.

Such fields or currents f are determined as an approximated solution, which is the series of known functions f_n (named basis functions) and unknown coefficients a_n

$$f(\boldsymbol{r}) = \sum_{n=1}^{N} a_n \, f_n(\boldsymbol{r})$$

a_n = unknown scalar coefficients

f_n = basis functions (either scalar or vector functions)

FORMULATION OF THE PROBLEM

We consider the deterministic functional equation

$$L(f) = g$$

L is a linear integro-differential operator (known)

f denoted the unknown function to determine (field or current)

g is a known function (excitation of the system,
e.g., antenna feeding)

EXAMPLE 1

A simple example of a functional equation is:

$$-\frac{d^2 f}{dx^2} = g(x) \qquad 0 \le x \le 1$$

$$f(0) = f(1) = 0$$

- The linear operator *L* is given by $-\frac{d^2}{dx^2}$ and the boundary conditions
- *f* represents the unknown function to determine
- *g* is a known function

EXAMPLE 2

This second example is more related to em problems:

$$\boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{1}{j\omega\mu\varepsilon}\nabla\nabla\cdot\boldsymbol{A} =$$

$$= \left(-j\omega + \frac{1}{j\omega\mu\varepsilon}\nabla\nabla\cdot\right)\left(\frac{\mu}{4\pi}\int_V \frac{e^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{J}(\boldsymbol{r}')\,dV'\right)$$

- The linear operator *L* is much more complex
- ***J*** is the unknown function to determine (current density)
- ***E*** is a known function (given electric field)

LINEAR OPERATOR

The operator *L* is a linear operator, which satisfies the relation:

$$L(c_1 f + c_2 h) = c_1 L(f) + c_2 L(h)$$

In most electromagnetic problems, the operator *L* is usually an integral, differential, o integro-differential operator.

$$\int_s \underline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')\, dS = \boldsymbol{E}_0(\boldsymbol{r})$$

The operator *L* sets a relation between the fields or currents and the excitation inside the investigation domain.

LINEAR OPERATOR

The inner product of two functions *u* e *v*, denoted by $< u, v >$, is a scalar quantity, which meets the following properties:

$$< u, v > = < v, u >$$ (commutative property)

$$< \alpha u + \beta v, h > = \alpha < u, h > + \beta < v, h >$$ (distributive property)

$$< u, u^* > \; > 0 \qquad \text{if } u \neq 0$$

$$< u, u^* > \; = 0 \qquad \text{if } u = 0$$

Examples of inner product:

$$< u(x), v(x) > = \int_C u(x)\, v(x)\, d\ell$$ (*u* and *v* scalar functions)

$$< \boldsymbol{u}(x), \boldsymbol{v}(x) > = \int_C \boldsymbol{u}(x) \cdot \boldsymbol{v}(x)\, d\ell$$ (***u*** and ***v*** vector functions)

APPLICATION OF THE MOM / 1

The MoM provides approximate solutions of integro-differential equations

$$L(f) = g$$

where L is a linear operator, f a unknown function which satisfies some boundary conditions, and g a known function.

The application of the MoM requires the following steps:

1. The unknown function f is approximated as a linear combination of N known functions f_n (basis functions)

$$f(\boldsymbol{r}) \cong \sum_{n=1}^{N} a_n f_n(\boldsymbol{r})$$

where a_n are unknown scalar coefficients.

APPLICATION OF THE MOM / 2

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2. The approximated representation of the unknown functions f is replaced in the equation, and the linearity of the operator L is exploited:

$$L\left(\sum_{n=1}^{N} a_n\, f_n\right) = \sum_{n=1}^{N} a_n\, L(f_n) = g$$

In this way, we get an equation with *N* scalar unknown coefficients.

The new task is the determination of the *N* coefficients a_n.

⇨ *N* scalar coefficients need to be determined.

Application of the MoM / 3

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3. The problem solution requires to formulate N independent equations:

- by imposing that the equation is satisfied exactly in N points

$$\sum_{n=1}^{N} a_n \, L\big(f_n(\boldsymbol{r}_m)\big) = g(\boldsymbol{r}_m) \qquad (m = 1..N)$$

- by imposing that the equation is satisfied according to weighted averages

$$\sum_{n=1}^{N} a_n \langle w_m, L\big(f_n\big) \rangle = \langle w_m, g \rangle \qquad (m = 1..N)$$

where w_m are the test functions (often identical to basis functions)

APPLICATION OF THE MOM / 4

Some comments:

a. The usually adopted inner product is

$$<\boldsymbol{u},\boldsymbol{v}>=\int_S \boldsymbol{u}\cdot\boldsymbol{v}\,dS$$

b. Imposing the first condition does not allow verifying the equation in the entire domain, but only locally.

c. The second condition imposes that the equation holds "in average" in the entire domain.

d. The first condition is a special case of the second one, where the test functions are Dirac delta functions (point-matching technique).

e. If test functions coincide with basis functions, the method is denoted as the Galerkin method.

4. The resulting system of N equation can be recast in matrix form:

$$\begin{bmatrix} <w_1, L(f_1)> & <w_1, L(f_2)> & . & . \\ <w_2, L(f_1)> & <w_2, L(f_2)> & & \\ . & & & \\ . & & & \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \end{bmatrix} = \begin{bmatrix} <w_1, g> \\ <w_2, g> \\ . \\ . \end{bmatrix}$$

or, in a more compact form:

$$\mathbf{A}\,\mathbf{X} = \mathbf{B}$$

$\mathbf{A}$ square matrix NxN

$\mathbf{X}$ column vector Nx1

$\mathbf{B}$ column vector Nx1

5. The solution of the matrix equation permits to determine the vector $\mathbf{X}$ and consequently the unknown function f:

$$\mathbf{X} = \mathbf{A}^{-1}\,\mathbf{B}$$

APPLICATION OF THE MOM / SUMMARY

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The application of the MoM to the problem $L(f)=g$ requires the following steps:

1. Definition of the basis functions f_n $\qquad f(\boldsymbol{r})=\sum_{n=1}^{N} a_n\, f_n(\boldsymbol{r})$

2. Definition of the test functions w_m

3. Filling of matrix **A** and vector **B**

$$A_{mn} = < w_m , L(f_n) > \qquad\qquad B_m = < w_m , g >$$

4. Solution of the matrix equation $\mathbf{A\,X}=\mathbf{B}$

EXAMPLE / 1

Let us consider the following differential equation:

$$-\frac{d^2 f}{dx^2} = 1 + 4x^2 \qquad 0 \le x \le 1$$
$$f(0) = f(1) = 0$$

By using the notation previously introduced, the linear operator *L* results:

$$L(\cdot) = \begin{cases} -\dfrac{d^2}{dx^2}(\cdot) & 0 \le x \le 1 \\ f(0) = f(1) = 0 \end{cases}$$

and the excitation *g* is expressed as:

$$g(x) = 1 + 4x^2$$

EXAMPLE / 2

In the application of the MoM, we adopt polinomial basis and test functions:

$$f_n = w_n = x - x^{n+1}$$

Moreover, the inner product is defined as:

$$< f, g >= \int_0^1 f(x) g(x) \mathrm{d}x$$

NOTE: the analytical solution of this problem is known:

$$f = \frac{5}{6} x - \frac{1}{2} x^2 - \frac{1}{3} x^4$$

EXAMPLE / 3

case N=1

The only basis and test function is:

$$f_1 = w_1 = x - x^2$$

The matrix problem results:

$$[< w_1, L(f_1) >][a_1] = [< w_1, g >]$$

$$\left[\frac{1}{3}\right][a_1] = \left[\frac{11}{30}\right] \quad \Longrightarrow \quad a_1 = \frac{11}{10}$$

The approximate solution is:

$$f \cong a_1 f_1 = \frac{11}{10}\left(x - x^2\right)$$

EXAMPLE / 4

case N=2

The two basis and test functions are:

$$f_1 = w_1 = x - x^2 \qquad\qquad f_2 = w_2 = x - x^3$$

The matrix problem results:

$$\begin{bmatrix} < w_1, L(f_1) > & < w_1, L(f_2) > \\ < w_2, L(f_1) > & < w_2, L(f_2) > \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} < w_1, g > \\ < w_2, g > \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 1/2 \\ 1/2 & 4/5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 11/30 \\ 7/12 \end{bmatrix} \Rightarrow \quad a_1 = \frac{1}{10} \qquad a_2 = \frac{2}{3}$$

The approximate solution is:

$$f \cong a_1 f_1 + a_2 f_2 = \frac{23}{30} x - \frac{1}{10} x^2 - \frac{2}{3} x^3$$

EXAMPLE / 5

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case N=3

The three basis and test functions are:

$$f_1 = w_1 = x - x^2 \qquad f_2 = w_2 = x - x^3 \qquad f_3 = w_3 = x - x^4$$

The matrix problem results:

$$\begin{bmatrix} <w_1, L(f_1)> & <w_1, L(f_2)> & <w_1, L(f_3)> \\ <w_2, L(f_1)> & <w_2, L(f_2)> & <w_2, L(f_3)> \\ <w_3, L(f_1)> & <w_3, L(f_2)> & <w_3, L(f_3)> \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} <w_1, g> \\ <w_2, g> \\ <w_3, g> \end{bmatrix}$$

$$\begin{bmatrix} 1/3 & 1/2 & 3/5 \\ 1/2 & 4/5 & 1 \\ 3/5 & 1 & 9/7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 11/30 \\ 7/12 \\ 51/70 \end{bmatrix} \Rightarrow a_1 = \frac{1}{2} \qquad a_2 = 0 \qquad a_3 = \frac{1}{3}$$

The resulting (**exact!**) solution is:

$$f \cong a_1 f_1 + a_2 f_2 + a_3 f_3 = \frac{5}{6} x - \frac{1}{2} x^2 - \frac{1}{3} x^4$$

EXAMPLE / 6

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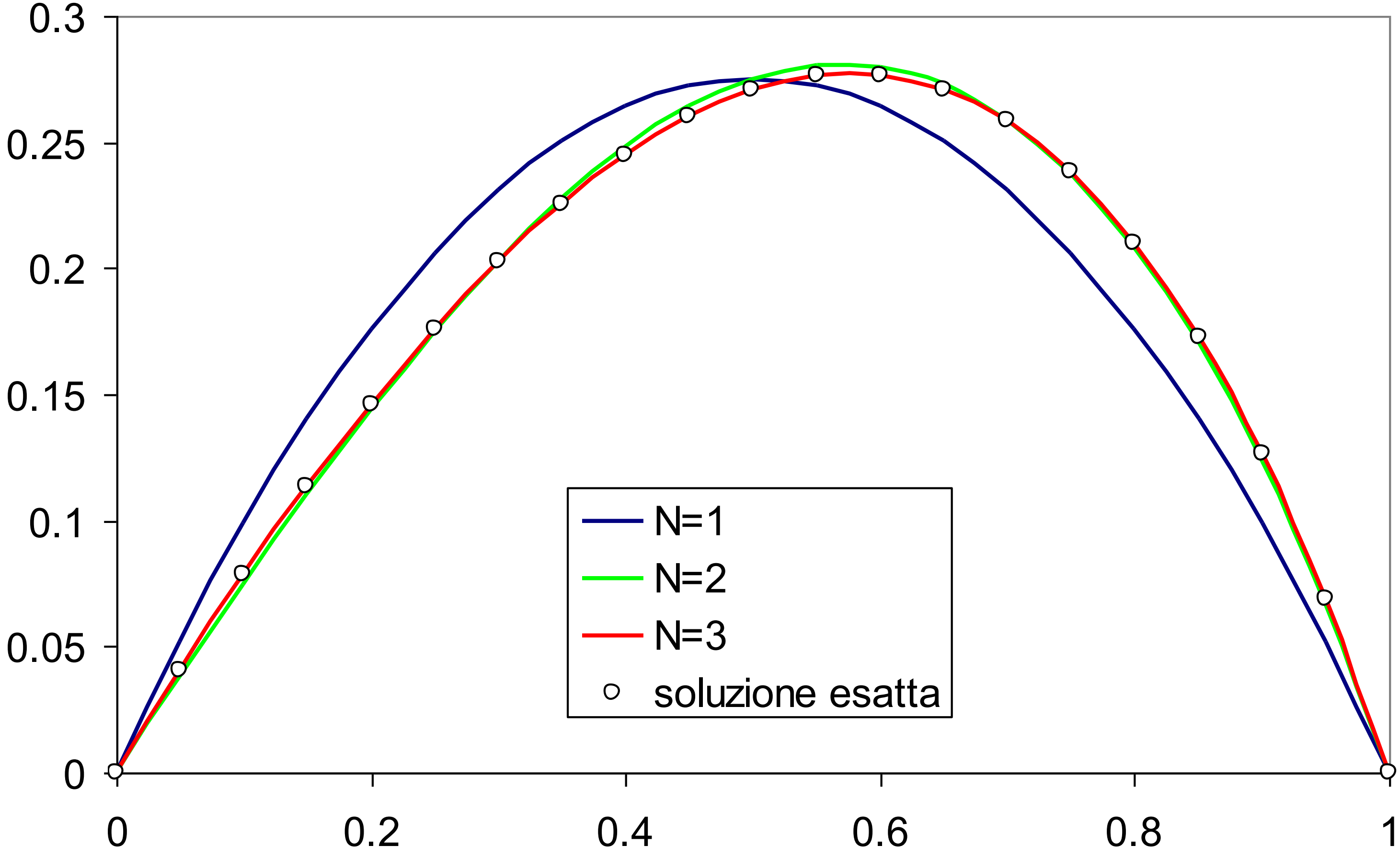

The exact solution is obtained with N=3 basis functions.