Lecture 8

THE BOUNDARY ELEMENT METHOD (BEM)

INTRODUCTION

University of Pavia

The boundary-element method (BEM) is an efficient numerical technique, adopted for the solution of large classes electromagnetic problems.

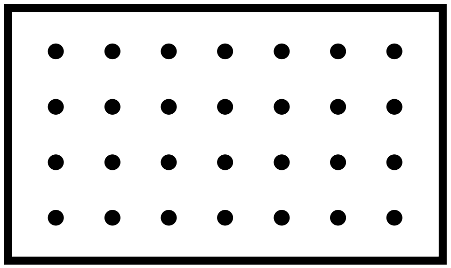
The BEM is based on the formulation of the problem in terms of an integral equation, which is solved by the Method of Moments (MoM).
It is typically formulated in the frequency domain, and it can be applied to either closed or open problems, typically filled with a homogeneous or stratified dielectric medium.

Different names are sometimes adopted for the BEM:

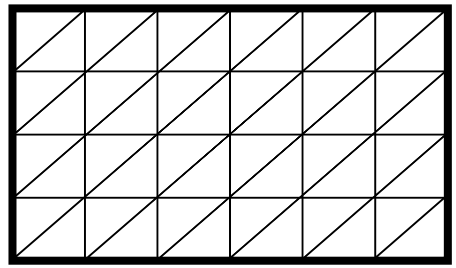
- Integral Equation Method (IEM)
- Boundary Integral Method (BIM)
- Method of Moments (MoM)

INTRODUCTION

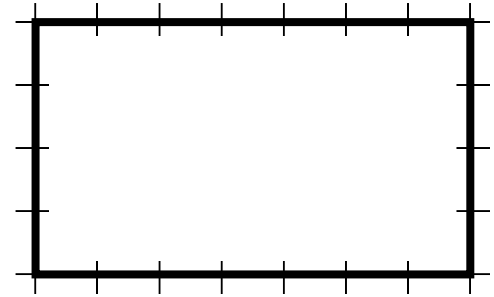
The BEM permits to **reduce the dimensionality of the problem**, because the unknown function is defined on the discontinuities only.



FDTD segmentation



FEM segmentation



BEM segmentation

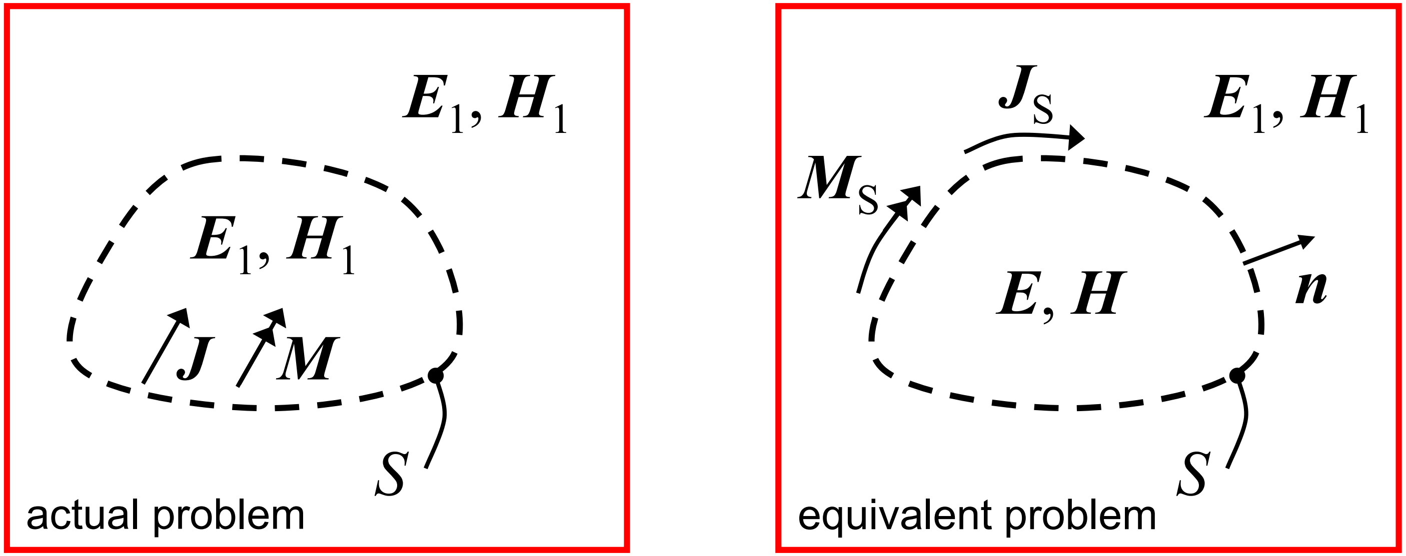
The unknown function is typically a current density, related to the fields through the Green’s function. In the case of open problems, the radiation condition is included in the Green’s function (no ABC or PML required).

Reference books:

1. R. F. Harrington, *Field Computation by Moment Methods*, IEEE Press, 1993.
2. W. C. Chew *et al.*, *Integral Equation Methods for Electromagnetics and Elastic Waves*, Morgan & Claypool Publishers, 2008.

EQUIVALENCE THEOREM

The equivalent field theorem says that actual sources (such as an antenna or a scatterer) can be replaced by equivalent sources which produce the same field within a region.



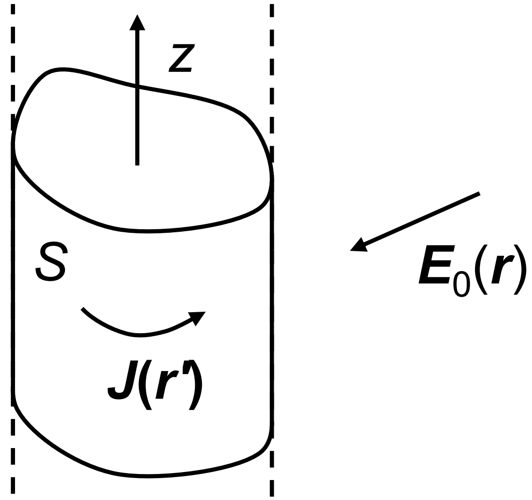
The equivalent current densities are:

$$J_S = \hat{n} \times (H_1 - H) \qquad M_S = -\hat{n} \times (E_1 - E)$$

EXAMPLE

Let us consider a conductive cylinder illuminated by a known incident electric field $\boldsymbol{E}_0$ (excitation of the system). The scattered field $\boldsymbol{E}_{\text{scat}}$ is the effect of an equivalent current density $\boldsymbol{J}$ (problem unknown), on the surface of S of the cylinder, expressed through a Green's integral:

$$\boldsymbol{E}_{\text{scat}}(\boldsymbol{r}) = \int_S \underline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')\, dS'$$



where $\underline{\boldsymbol{G}}$ is the Green's function the electric current density to the electric field.

The integral equation is obtained by imposing the electric wall condition on the surface S of the metal cylinder.

$$\mathbf{n}\times\mathbf{E}_{\text{scat}}(\mathbf{r}) + \mathbf{n}\times\mathbf{E}_0(\mathbf{r}) = 0 \quad \text{on } S$$

$$\Rightarrow \quad \mathbf{n}\times\int_S \underline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\, dS' = -\mathbf{n}\times\mathbf{E}_0(\mathbf{r})$$

PHYSICAL DEFINITION

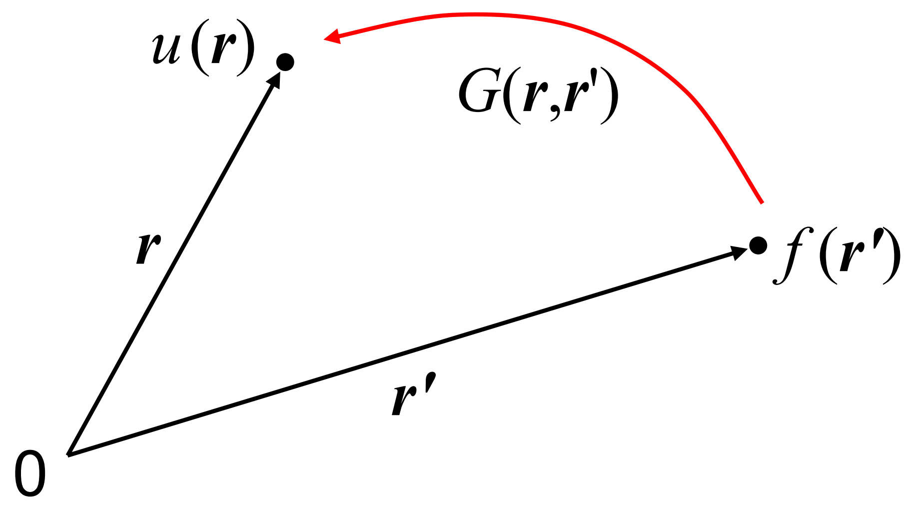
Green's function $G(\boldsymbol{r},\boldsymbol{r}')$ represents the response in a point $\boldsymbol{r}$ (field point) determined by a point source located in $\boldsymbol{r}'$ (source point).

point source

$$f(\boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}')$$

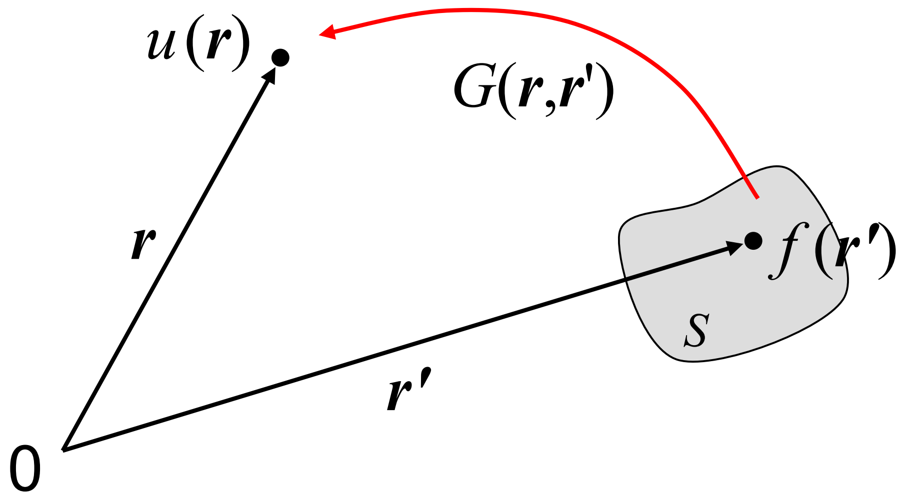
resulting field

$$u(\boldsymbol{r}) = G(\boldsymbol{r},\boldsymbol{r}')$$



If the source point is replaced by a distributed source, the resulting field is the obtained by superimposing the point source responses.

$$u(\boldsymbol{r}) = \int_S G(\boldsymbol{r},\boldsymbol{r}')\, f(\boldsymbol{r}')\, dS'$$



If the source is $f(\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}')$, it is straightforward to show that

$$u(\boldsymbol{r}) = \int_S G(\boldsymbol{r},\boldsymbol{r}')\, \delta(\boldsymbol{r}-\boldsymbol{r}')\, dS' = G(\boldsymbol{r},\boldsymbol{r}')$$

MATHEMATICAL DEFINITION

The Green's function $G(\boldsymbol{r},\boldsymbol{r}')$ provides a systematic technique to transform an differential problem into an integral problem.

Let us consider the problem

$$L\big(u(\boldsymbol{r})\big) = f(\boldsymbol{r})$$

where

- L is a linear differential operator
- $u(\boldsymbol{r})$ is the unknown function
- $f(\boldsymbol{r})$ is a given function (that represents the excitation).

The unknown function $u(\mathbf{r})$ can be formally expressed as

$$u(\mathbf{r}) = L^{-1}\left(f(\mathbf{r})\right)$$

where L^{-1} represents the inverse of operator L.

Since L is differential, its inverse operator L^{-1} is an integral operator

$$u(\mathbf{r}) = L^{-1}\left(f(\mathbf{r})\right) = \int_S G(\mathbf{r},\mathbf{r}')\, f(\mathbf{r}')\, dS'$$

where $G(\mathbf{r},\mathbf{r}')$ is the Green's function associated to operator L.

1. The Green's function $G(\boldsymbol{r},\boldsymbol{r}')$ satisfies the relation:

$$L\big(G(\boldsymbol{r},\boldsymbol{r}')\big)=\delta(\boldsymbol{r}-\boldsymbol{r}')$$

where $\delta(\boldsymbol{r}-\boldsymbol{r}')$ represents the delta function.

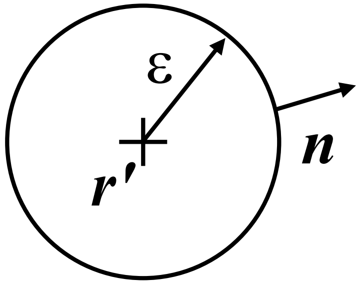
2. The Green's function $G(\boldsymbol{r},\boldsymbol{r}')$ is symmetric, so that:

$$G(\boldsymbol{r},\boldsymbol{r}')=G(\boldsymbol{r}',\boldsymbol{r})$$

3. The Green's function $G(\boldsymbol{r},\boldsymbol{r}')$ satisfies the boundary conditions of the associated linear operator L for function $u(\boldsymbol{r})$.

4. The partial derivative $\partial G/\partial n$ is discontinuous in $\boldsymbol{r}'$:

$$\lim_{\varepsilon\to 0}\oint_S \frac{\partial G}{\partial n}\, dS = 1$$



GENERAL OBSERVATIONS:

Each Green's function is associated to a given differential equation with its boundary conditions.

The determination of the Green's function sometimes requires a significant effort, especially for its derivation in closed form or in the form of a rapidly converging series.

Some electromagnetic problems can be formulated in terms of **scalar Green's functions**, other problems require the use of **dyadic Green's functions**.

FREE SPACE GREEN'S FUNCTION / 1

We calculate the Greens' function associated to the following partial differential equation:

$$\nabla^2\Phi = g \qquad \text{(boundary condition } \Phi = f\text{)}$$

The Green's function $G(\boldsymbol{r},\boldsymbol{r}')$ must satisfy the relation:

$$\nabla^2 G(\boldsymbol{r},\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}') \qquad \text{(boundary condition } G = f\text{)}$$

It is usually convenient to represent the Green's function $G(\boldsymbol{r},\boldsymbol{r}')$ as the sum of two terms:

$$G(\boldsymbol{r},\boldsymbol{r}') = F(\boldsymbol{r},\boldsymbol{r}') + U(\boldsymbol{r},\boldsymbol{r}') \qquad \begin{cases} \nabla^2 F(\boldsymbol{r},\boldsymbol{r}') = \delta(\boldsymbol{r}-\boldsymbol{r}') \\ \nabla^2 U(\boldsymbol{r},\boldsymbol{r}') = 0 \end{cases}$$

Function F is denoted as the free space Green's function.

Function U is selected to satisfy the boundary condition $U=f-F$.

FREE SPACE GREEN'S FUNCTION / 2

2D CASE

The 2D Laplace operator is $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

The Green's function $G(x,y,x',y')$ satisfies the relation:

$$\nabla^2 G(x,y,x',y') = \delta(x-x')\,\delta(y-y')$$

Therefore, function $F(x,y,x',y')$ must satisfy the same relation:

$$\nabla^2 F(x,y,x',y') = \delta(x-x')\,\delta(y-y')$$

In polar coordinates, for $\rho = \sqrt{(x-x')^2 + (y-y')^2} > 0$, it results:

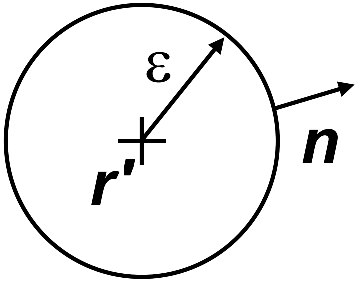
$$\nabla^2 F = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial F}{\partial \rho}\right) = 0$$

FREE SPACE GREEN'S FUNCTION / 3

After integrating twice, it gives:

$$F = A \ln \rho + B$$

Applying the 4th property of the Green's functions



$$\lim_{\varepsilon \to 0} \oint \frac{dF}{d\rho} d\ell = \lim_{\varepsilon \to 0} \int_0^{2\pi} \frac{A}{\rho} \rho \, d\phi = 2\pi A = 1$$

thus resulting $A=1/2\pi$. Since B is arbitrary, we may choose $B=0$. Thus

$$F = \frac{1}{2\pi} \ln \rho$$

And finally:

$$\boxed{G = F + U = \frac{1}{2\pi} \ln \rho + U}$$

We choose U so that G satisfies prescribed boundary conditions.

3D CASE

The 3D Laplace operator is $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

The Green's function $G(x,y,x',y')$ satisfies the relation:

$$\nabla^2 G(x,y,z,x',y',z') = \delta(x-x')\,\delta(y-y')\,\delta(z-z') = \delta(\boldsymbol{r}-\boldsymbol{r}')$$

Therefore, function $F(x,y,x',y')$ must satisfy the same relation:

$$\nabla^2 F(x,y,z,x',y',z') = \delta(x-x')\,\delta(y-y')\,\delta(z-z') = \delta(\boldsymbol{r}-\boldsymbol{r}')$$

In spherical coordinates, for $r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} > 0$

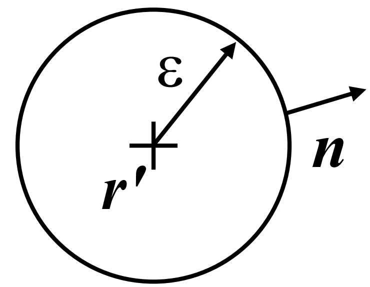
$$\nabla^2 F = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dF}{dr}\right) = 0$$

Free Space Green's Function / 5

After integrating twice, it gives:

$$F = -\frac{A}{r} + B$$

Applying the 4th property of the Green's functions



$$\lim_{\varepsilon \to 0} \oint \frac{dF}{dr} dS = \lim_{\varepsilon \to 0} \int_0^{2\pi} d\phi \int_0^{\pi} \frac{A}{r^2} r^2 d\theta = 4\pi A = 1$$

thus resulting $A=1/4\pi$. Since B is arbitrary, we may choose $B=0$. Thus

$$F = -\frac{1}{4\,\pi\,r}$$

And finally:

$$\boxed{G = F + U = -\frac{1}{4\,\pi\,r} + U}$$

We choose U so that G satisfies prescribed boundary conditions.

GF for Closed Domains / 1

The eigenfunction expansion is a technique for the analytical determination of the Green's function in domains with conducting boundaries, for differential equations whose homogeneous solution is known.

To illustrate the eigenfunction expansion procedure, we consider the Green's function for the wave equation

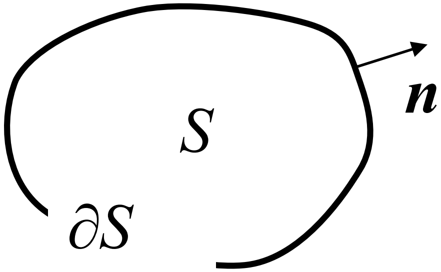
$$\nabla^2\Psi + k^2\Psi = 0 \qquad \text{(boundary condition } \Psi=0 \text{ or } \frac{\partial \Psi}{\partial n} = 0\text{)}$$

The eigenfunctions and eigenvalues $\{\Psi_j, k_j\}$ of this problem satisfy

$$\nabla^2\Psi_j + k_j^2\Psi_j = 0 \qquad \text{(boundary condition } \Psi_j=0 \text{ or } \frac{\partial \Psi_j}{\partial n} = 0\text{)}$$

and form a complete set of orthonormal functions:

$$\int_S \Psi_j \Psi_i^* dS = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$



GF FOR CLOSED DOMAINS / 2

The Green's function can be expanded in terms of eigenfunctions Ψ_j:

$$G(x, y, x', y') = \sum_{j=1}^{\infty} a_j \, \Psi_j(x, y)$$

Since the Green's function must satisfy

$$\nabla^2 G + k^2 G = \delta(x - x')\,\delta(y - y')$$

we obtain

$$\sum_{j=1}^{\infty} a_j \left(\nabla^2 \Psi_j + k^2 \Psi_j\right) = \delta(x - x')\,\delta(y - y')$$

By applying the relation $\nabla^2 \Psi_j + k_j^2 \Psi_j = 0$, we obtain

$$\sum_{j=1}^{\infty} a_j \,(k^2 - k_j^2)\,\Psi_j = \delta(x - x')\,\delta(y - y')$$

GF FOR CLOSED DOMAINS / 3

Multiplying both sides by Ψ_i^* and integrating over the region S gives

$$\sum_{j=1}^{\infty} a_j (k^2 - k_j^2) \int_S \Psi_j \Psi_i^* \, dS = \Psi_i^*(x', y')$$

Imposing the orthonormal property or eigenfunctions leads to

$$a_i = \frac{\Psi_i^*(x', y')}{(k^2 - k_i^2)}$$

The Green's functions results:

$$G(x, y, x', y') = \sum_{j=1}^{\infty} \frac{\Psi_j(x, y) \Psi_j^*(x', y')}{(k^2 - k_j^2)}$$

Dyadic Green's Function / 1

The Green's functions considered so far can be adopted for the solution of electromagnetic problems described by **scalar partial differential equations**.

In the case of **vector equations** (e.g.., vector wave equation)

$$L\left(\boldsymbol{U}(\boldsymbol{r})\right)=\boldsymbol{F}(\boldsymbol{r})$$

Dyadic Green's functions are needed:

$$\boldsymbol{U}(\boldsymbol{r})=\int_S \underline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{F}(\boldsymbol{r}')\,\mathrm{d}S'$$

DYADIC GREEN'S FUNCTION / 2

In a Cartesian coordinate system, the dyadic Green's function can be expressed as

$$\underline{\boldsymbol{G}}(x,y,z,x',y',z') = G_{xx}\boldsymbol{xx}'+G_{xy}\boldsymbol{xy}'+G_{xz}\boldsymbol{xz}'+G_{yx}\boldsymbol{yx}'+G_{yy}\boldsymbol{yy}'+G_{yz}\boldsymbol{yz}'+G_{zx}\boldsymbol{zx}'+G_{zy}\boldsymbol{zy}'+G_{zz}\boldsymbol{zz}'$$

The dot product between a dyadic function and a vector returns a vector:

$$\underline{\boldsymbol{G}}(x,y,z,x',y',z')\cdot\boldsymbol{F}(x',y',z') = \begin{bmatrix} G_{xx}\boldsymbol{xx}' & G_{xy}\boldsymbol{xy}' & G_{xz}\boldsymbol{xz}' \\ G_{yx}\boldsymbol{yx}' & G_{yy}\boldsymbol{yy}' & G_{yz}\boldsymbol{yz}' \\ G_{zx}\boldsymbol{zx}' & G_{zy}\boldsymbol{zy}' & G_{zz}\boldsymbol{zz}' \end{bmatrix} \cdot \begin{bmatrix} F_x\boldsymbol{x}' \\ F_y\boldsymbol{y}' \\ F_z\boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} (G_{xx}F_x + G_{xy}F_y + G_{xz}F_z)\boldsymbol{x} \\ (G_{yx}F_x + G_{yy}F_y + G_{yz}F_z)\boldsymbol{y} \\ (G_{zx}F_x + G_{zy}F_y + G_{zz}F_z)\boldsymbol{z} \end{bmatrix}$$

DYADIC GREEN'S FUNCTION / 3

The free space scalar Green's function, associated to the wave equation

$$\nabla^2 \boldsymbol{A} + k^2 \boldsymbol{A} = -\mu \boldsymbol{J}$$

is given by

$$G_0(\boldsymbol{r}, \boldsymbol{r}') = \frac{\mathrm{e}^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

Therefore, the vector potential ***A*** is given by:

$$\boldsymbol{A}(\boldsymbol{r}) = \mu \int_V G_0(\boldsymbol{r}, \boldsymbol{r}')\, \boldsymbol{J}(\boldsymbol{r}')\, \mathrm{d}V' = \mu \int_V \frac{\mathrm{e}^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{J}(\boldsymbol{r}')\, \mathrm{d}V'$$

The electric field ***E*** can be expressed in terms of the vector potential ***A***

$$\boldsymbol{E}(\boldsymbol{r}) = -j\omega\left[1 + \frac{1}{k^2}\nabla\nabla\cdot\right]\boldsymbol{A}(\boldsymbol{r}) = -j\omega\mu\left[1 + \frac{\nabla\nabla\cdot}{k^2}\right]\int_V \frac{\mathrm{e}^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{J}(\boldsymbol{r}')\, \mathrm{d}V'$$

DYADIC GREEN'S FUNCTION / 4

By using the property $\nabla \cdot (a\,\boldsymbol{B}) = a\,\nabla \cdot \boldsymbol{B} + \nabla a \cdot \boldsymbol{B}$, it results that:

$$\nabla \cdot \frac{\mathrm{e}^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{J}(\boldsymbol{r}') = \cancel{\frac{\mathrm{e}^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\nabla \cdot \boldsymbol{J}(\boldsymbol{r}')} + \nabla \frac{\mathrm{e}^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \cdot \boldsymbol{J}(\boldsymbol{r}')$$

Moreover, by applying the relation:

$$\nabla \frac{\mathrm{e}^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} = -\nabla' \frac{\mathrm{e}^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}$$

It finally results:

$$\boldsymbol{E}(\boldsymbol{r}) = -j\omega\mu \int_V \underbrace{\left[\left(\underline{\boldsymbol{I}} - \frac{1}{k^2}\nabla\nabla'\right)\frac{\mathrm{e}^{-jk|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\right]}_{\underline{\boldsymbol{G}}_0(\boldsymbol{r},\boldsymbol{r}')} \cdot \boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}V'$$

where $\underline{\boldsymbol{G}}_0(\boldsymbol{r},\boldsymbol{r}')$ is the dyadic free space Green's function.