Lecture 9

APPLICATIONS OF THE BOUNDARY ELEMENT METHOD (BEM)

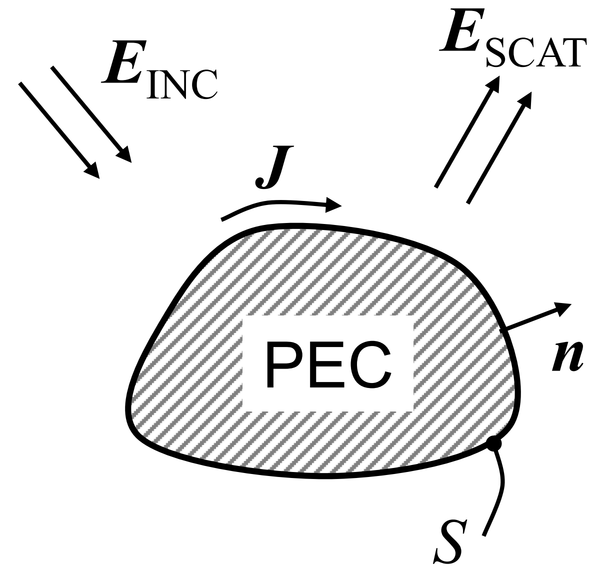
OUTLINE

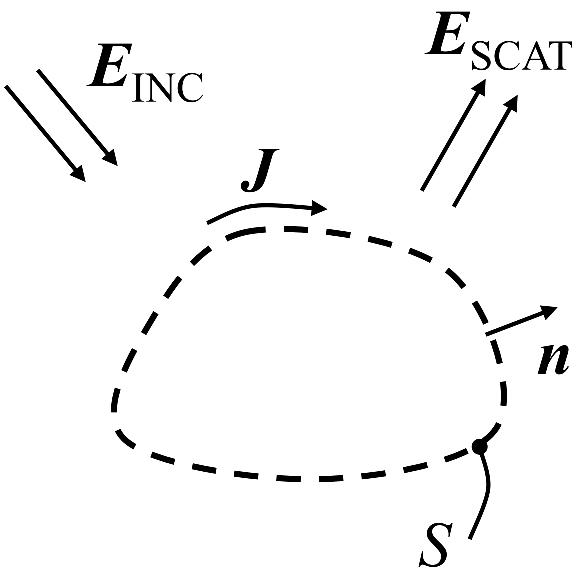
Three applications of the Boundary Element Method are presented in this lecture:

1. Scattering problem
2. Calculation of cavity modes
3. Modeling of a waveguide component

SCATTERING PROBLEM / 1

To simply see how an integral equation can arise physically, let us consider a scattering problem involving an arbitrarily shaped metallic scatterer (PEC), illuminated by a given electromagnetic wave $\boldsymbol{E}_{\mathrm{INC}}$. The first phenomenon that happens is that electric current $\boldsymbol{J}$ will be induced on the scatterer.





The current produces an electric field that exactly cancels the incident electric field inside the scatterer, as no electric field should exist inside. In addition, the current $\boldsymbol{J}$ also generates an electromagnetic field outside the scatterer yielding a scattered field $\boldsymbol{E}_{\mathrm{SCAT}}$.

SCATTERING PROBLEM / 2

The scattered field $\boldsymbol{E}_{\mathrm{SCAT}}$ can be expressed as a Green's integral:

$$\boldsymbol{E}_{\mathrm{SCAT}}(\boldsymbol{r}) = \int_S \underline{\boldsymbol{G}}_{\mathrm{EJ}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}S$$

where $\underline{\boldsymbol{G}}_{\mathrm{EJ}}(\boldsymbol{r},\boldsymbol{r}')$ is the free-space dyadic Green's function and $\boldsymbol{J}$ is the (unknown) electric current density on the surface S of the scatterer.

The total electric field must satisfy the electric wall condition on the surface S

$$\boldsymbol{n} \times \boldsymbol{E}_{\mathrm{TOT}} = \boldsymbol{n} \times \boldsymbol{E}_{\mathrm{INC}} + \boldsymbol{n} \times \boldsymbol{E}_{\mathrm{SCAT}} = 0 \qquad \text{on } S$$

This boundary condition leads to the integral equation:

$$\boldsymbol{n} \times \boldsymbol{E}_{\mathrm{INC}}(\boldsymbol{r}) + \boldsymbol{n} \times \int_S \underline{\boldsymbol{G}}_{\mathrm{EJ}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}S = 0 \quad \text{on } S$$

where the unknown is the electric current density $\boldsymbol{J}$ on the surface S. This integral equation is also known as the electric field integral equation (EFIE).

SCATTERING PROBLEM / 3

The resulting integral equation is solved by the MoM. The unknown function $\boldsymbol{J}$ is expressed as a combination of **vector basis functions**

$$\boldsymbol{J}(\boldsymbol{r}) \cong \sum_{n=1}^{N} I_n \, \boldsymbol{J}_n(\boldsymbol{r})$$

After substituting in the integral equation and applying the test with test functions $\boldsymbol{w}_{\mathrm{m}}=\boldsymbol{J}_{\mathrm{m}}$ and the inner product defined as a surface integral, it results

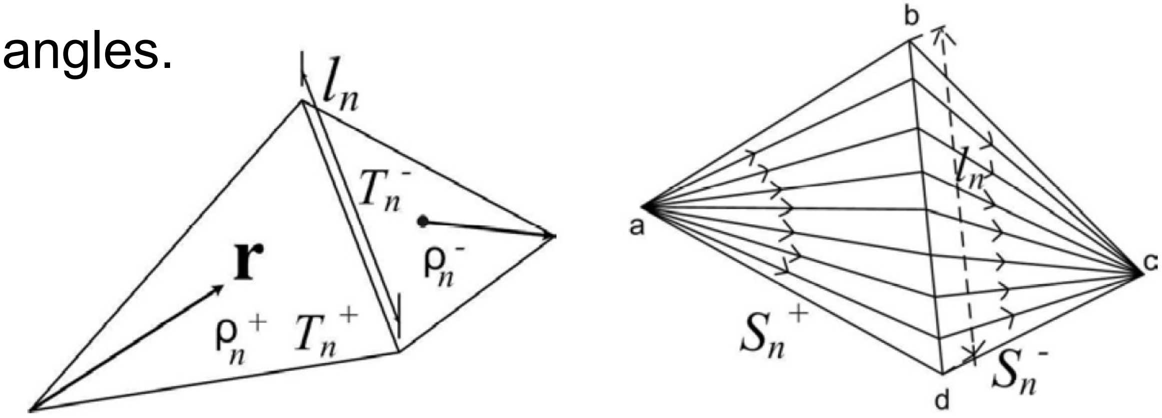
$$\sum_{n=1}^{N} I_n \underbrace{\int_S \boldsymbol{J}_m(\boldsymbol{r}) \cdot \int_S \underline{\boldsymbol{G}}_{\mathrm{EJ}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}_n(\boldsymbol{r}')\, \mathrm{d}S' \, \mathrm{d}S}_{Z_{mn}} = \underbrace{-\int_S \boldsymbol{J}_m(\boldsymbol{r}) \cdot \boldsymbol{E}_{\mathrm{INC}}(\boldsymbol{r})\, \mathrm{d}S}_{V_m} \quad (m = 1..N)$$

The system of equations can be recast in matrix form

$$\boldsymbol{Z}\,\boldsymbol{I} = \boldsymbol{V}$$

The BEM solution of the problem requires the choice of basis functions, the calculation of surface integrals, and the solution of the matrix equation.

In the application of the BEM to 3D problems, a popular choice of basis function is the **Rao-Wilton-Glisson (RWG) function**, defined on a pair of contiguous triangles.



The expression for the basis function describing the current on the triangle is

$$\boldsymbol{J}_n^s(\boldsymbol{r}) = \begin{cases} \dfrac{l_n}{2A_n^\pm}\rho_n^\pm & \boldsymbol{r} \in T_n^\pm \\ 0 & \text{otherwise} \end{cases}$$

where ± denotes the respective triangles, l_n is the edge length of the edge between the two triangles, A_n is the area of the triangles, $\rho_n(\boldsymbol{r})$ is the vector from the point $\boldsymbol{r}$ to the triangle apex, and T_n is the support of the triangles.

SURFACE INTEGRATION

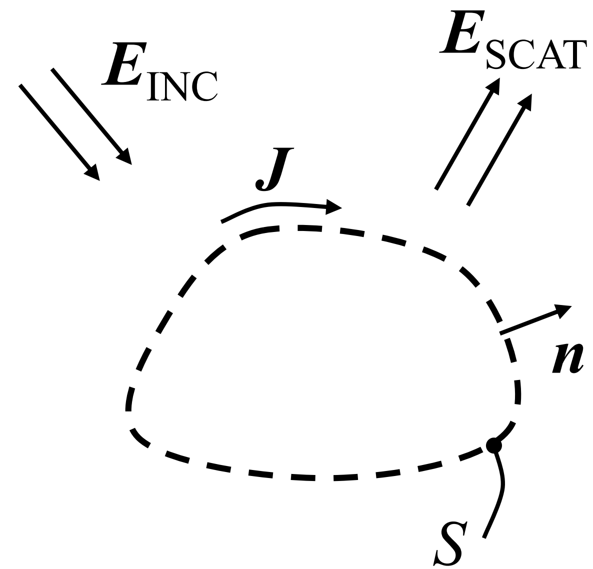
The surface integration is required for the calculation of the MoM matrix. In most cases, the double surface integration is performed numerically, by transforming the integral into the summation of weighted samples of the integrand function:

$$\int_a^b f(x)\,dx \cong \sum_{i=1}^{N} \omega_i \, f(x_i)$$

where x_i are the points where the function is calculated, and ω_i are the corresponding weights (e.g., by the Gauss method).

When the basis function and the test function coincide (elements on the matrix diagonal), the Green's function exhibits a **singularity**, which needs to be integrated analytically or with special techniques.

INTERNAL RESONANCES



Under particular circumstances, it can happen that the MoM matrix is singular for some values of frequency:

$$\boldsymbol{Z}\,\boldsymbol{I} = \boldsymbol{V}$$

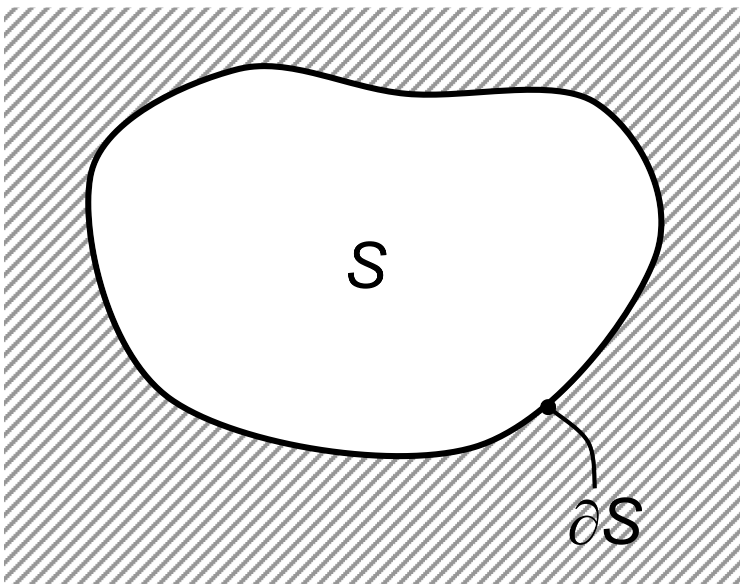
This behavior has a physical reason, which is related to internal resonances of the cavity with electric-wall condition at the boundary S.

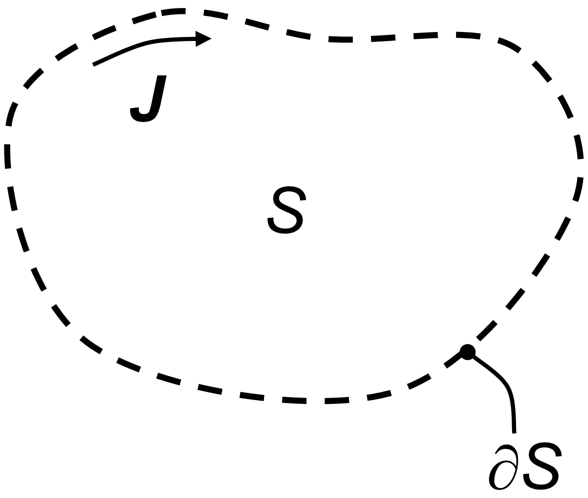
This phenomenon brings two effects:

- It suggests a technique to calculate the cavity modes
- It causes erroneous results in the solution of the scattering problem, thus requiring alternative approaches

Calculation of Cavity Modes

The cavity modes (or proper modes) are field patterns, which can exist in the absence of any excitation.





CALCULATION OF CAVITY MODES

The electromagnetic problem can be formulated in a way similar to the scattering problem, with no excitation:

$$\boldsymbol{n} \times \boldsymbol{E}_{\mathrm{SCAT}}(\boldsymbol{r}) = \boldsymbol{n} \times \int_{\partial S} \underline{\boldsymbol{G}}_{\mathrm{EJ}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}\ell = 0 \quad \text{su } \partial S$$

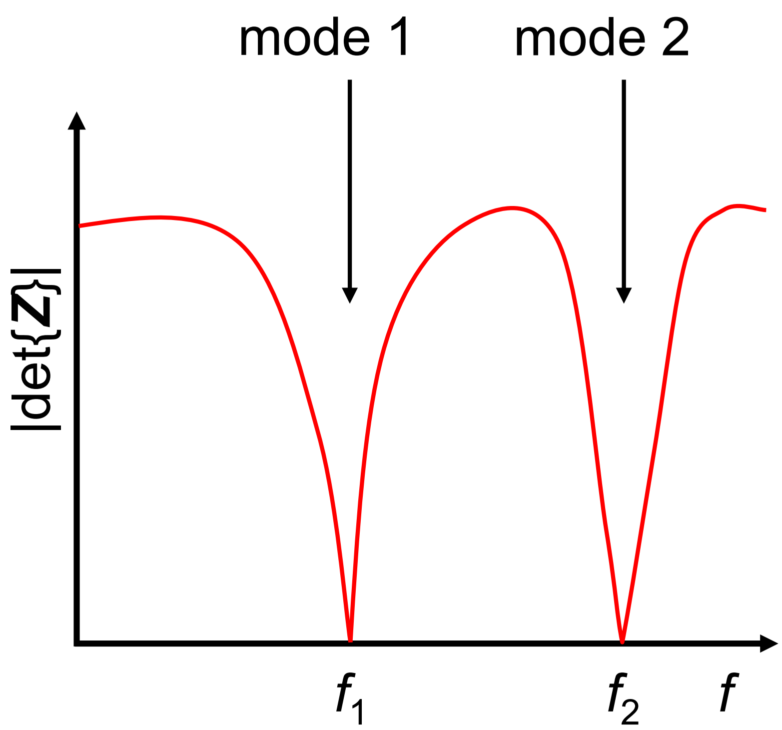
After applying the MoM, we obtain the following homogeneous matrix equation:

$$\boldsymbol{Z}(f)\,\boldsymbol{I} = 0$$

The matrix problem admits non-trivial solution at some frequencies, which correspond to the resonance (or cutoff) frequencies of the proper modes.

The determination of the modes requires a fine frequency scan. The non-trivial solution are found at those frequencies where one or more eigenvalues of the Z matrix vanish.

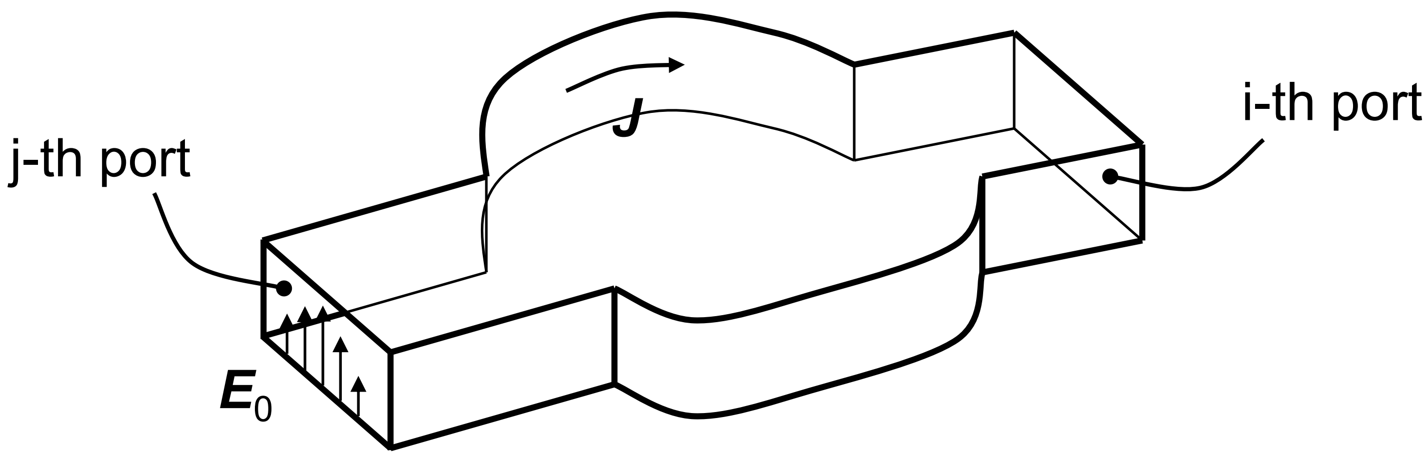
The eigenvectors associated to the null eigenvalues provide information on the modal current density on the cavity walls.



Calculation of the admittance matrix of an arbitrary planar waveguide component.

The generic entry (*i*,*j*) of the admittance matrix of the component is defined as:

$$Y_{ij} = \left. \frac{I_i}{V_j} \right|_{V_k = 0 \;\; \forall k \neq j}$$



MODELING OF A WAVEGUIDE CIRCUIT / 2

Following the definition of the admittance matrix, we impose the electric field at the j-th port:

$$\boldsymbol{E}_0(\boldsymbol{r}) = V_{\mathrm{j}}\, \boldsymbol{e}_{\mathrm{j}}(\boldsymbol{r})$$

where $\boldsymbol{e}_{\mathrm{j}}$ is the electric modal vector of the fundamental mode at the j-th port, and V_{j} represents the corresponding modal voltage.

NOTE Please note that $\boldsymbol{E}_0$ does not represent the incident field, but the total field at the j-th port.

By applying the equivalence theorem, the metal walls of the waveguide component are replaced by a surface electric current density $\boldsymbol{J}$, acting in free space.

MODELING OF A WAVEGUIDE CIRCUIT / 3

The current $\boldsymbol{J}$ determines an electric field $\boldsymbol{E}_{\mathrm{SCAT}}$, which can be expressed in the following form

$$\boldsymbol{E}_{\mathrm{SCAT}}(\boldsymbol{r}) = \int_S \underline{\boldsymbol{G}}_{\mathrm{EJ}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}S$$

where $\underline{\boldsymbol{G}}_{\mathrm{EJ}}(\boldsymbol{r},\boldsymbol{r}')$ represents the free space Green's function.

The electric field $\boldsymbol{E}_{\mathrm{SCAT}}$ must satisfy the electric wall condition on the surface S of the component, except on the j-th port, where the value of the electric field is $\boldsymbol{E}_0$:

$$\boldsymbol{n} \times \boldsymbol{E}_{\mathrm{SCAT}} = \begin{cases} \boldsymbol{E}_0 & \text{on the j-th port} \\ 0 & \text{elsewhere on } S \end{cases}$$

MODELING OF A WAVEGUIDE CIRCUIT / 4

The resulting integral equation can be solved by using the Method of Moments, and it allows to determine the current ***J***.

Once the current ***J***, is known, the magnetic field $\boldsymbol{H}_{\mathrm{SCAT}}$ everywhere in the component can be computed by the following expression:

$$\boldsymbol{H}_{\mathrm{SCAT}}(\boldsymbol{r}) = \int_S \underline{\boldsymbol{G}}_{\mathrm{HJ}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}S$$

where $\underline{\boldsymbol{G}}_{\mathrm{HJ}}(\boldsymbol{r},\boldsymbol{r}')$ is the free space Green's function relating the electric current density and the magnetic field.

Finally, the modal current I_i at the i-th port can be computed as

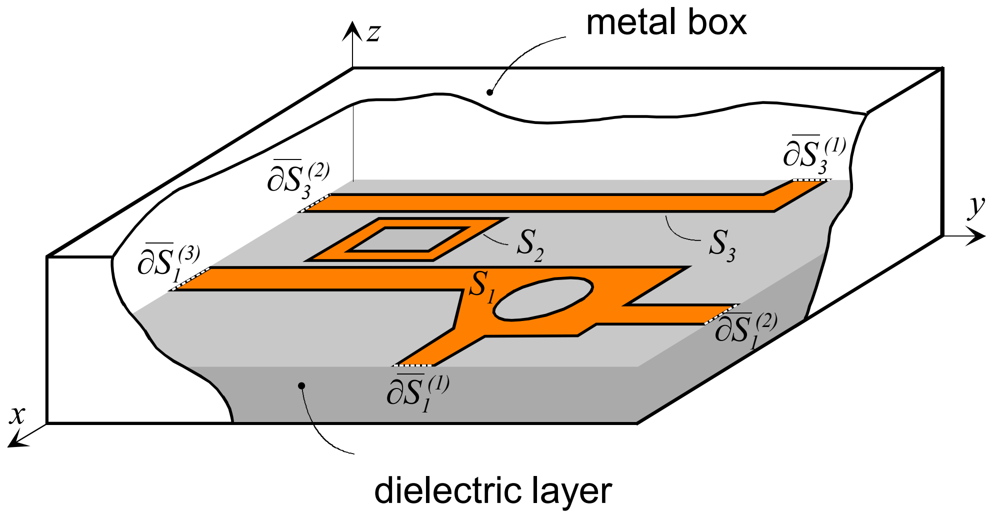
$$I_{\mathrm{i}} = \int_{S_{\mathrm{i}}} \boldsymbol{h}_{\mathrm{i}}(\boldsymbol{r}) \cdot \boldsymbol{H}_{\mathrm{SCAT}}(\boldsymbol{r})\,\mathrm{d}S = \int_{S_{\mathrm{i}}}\int_S \boldsymbol{h}_{\mathrm{i}}(\boldsymbol{r}) \cdot \underline{\boldsymbol{G}}_{\mathrm{HJ}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}S'\,\mathrm{d}S$$

If V_j is selected 1 V, the expression of the current I_i coincides with the (i,j) entry of the Y matrix.

SHIELDED MICROSTRIP CIRCUIT / 1

Calculation of the admittance matrix **Y** of a shielded microstrip circuit.

Similarly to the previous case of waveguide circuits, the ij-th entry of the admittance matrix is defined as:



$$Y_{ij} = \left. \frac{I_i}{V_j} \right|_{V_k=0 \ \forall k \neq j}$$

SHIELDED MICROSTRIP CIRCUIT / 2

The presence of the outer metal box does not interfere with the electromagnetic characteristics of the circuit, but it allows adopting a modal Green's function, whose calculation is straightforward and particularly suitable to the case of multilayered geometry.

NOTE. This technique is adopted in the commercial electrimagnetic solver AWR Microwave Office.

The ports of the circuit are defined by using the "delta gap" excitation. The electric field at the j-th port is:

$$\boldsymbol{E}_0(\boldsymbol{r}) = \delta(\boldsymbol{r} - \boldsymbol{r}_\mathrm{j})\, V_\mathrm{j}\, \boldsymbol{n}_\mathrm{j}$$

where $\boldsymbol{r}_\mathrm{j}$ is a point on the j-th port segment and $\boldsymbol{n}_\mathrm{j}$ is the (inner) unit normal vector of the j-th port.

SHIELDED MICROSTRIP CIRCUIT / 3

By applying the equivalence theorem, all conductors in the circuit (except the metal box) are replaced by a surface electric current density ***J***, acting inside the metal box.

The current ***J*** determines an electric field $\boldsymbol{E}_{\mathrm{SCAT}}$, which can be expressed as a Green's integral

$$\boldsymbol{E}_{\mathrm{SCAT}}(\boldsymbol{r}) = \int_S \underline{\boldsymbol{G}}_{\mathrm{EJ}}(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}S$$

where $\underline{\boldsymbol{G}}_{\mathrm{EJ}}(\boldsymbol{r},\boldsymbol{r}')$ represents the Green's function of the metal box, filled with the stratified dielectric medium.

The calculation of the admittance matrix is then performed as in the case of the waveguide circuits.