EE840 COMPUTATIONAL ELECTROMAGNETICS – FALL 2002

Lecture #5: Introduction to 1D FDTD

1 Key points from last lecture:

1. Began introducing 1D FDTD method.

2 Today's Objectives

1. Complete the introduction of the 1D FDTD method.

2.1 One Dimensional Formulation

In the formulation of the one-dimensional FDTD method we can assume TEM-polarization. That is to say that both the electric and magnetic fields are perpendicular to the direction of propagation. This results from the fact that in one-dimension we are basically modeling plane wave propagation (since the field cannot scatter in either of the other two directions), consequently, all of the partial derivatives except those with respect to the axis of propagation are zero. Therefore, Maxwell's equations, in a rectangular coordinate system, can be reduced to two coupled equations, and assuming propagation in the $\hat{x}$ direction, we have

$$\begin{gathered} \nabla \times E = -\mu \frac{\partial H}{\partial t} \quad and \quad \nabla \times H = \in \frac{\partial E}{\partial t} + J, or \\ \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \quad and \quad -\frac{\partial H_y}{\partial x} = \in \frac{\partial E_z}{\partial t} + \sigma E_z . \end{gathered} \tag{109}$$

Note that the minus sign for H_y results from the power flow in the $+\hat{x}$ direction. Using the difference formulations presented above we get

$$\frac{H_y^{n+1/2}(i) - H_y^{n-1/2}(i)}{\Delta t} = \frac{-1}{\mu}\left[\frac{E_z^n(i+1) - E_z^n(i)}{\Delta x}\right]. \tag{110}$$

Rearranging and solving for the most recent time value we have

$$H_y^{n+1/2}(i) = H_y^{n-1/2}(i) + \frac{-1}{\mu}\left[\frac{E_z^n(i+1) - E_z^n(i)}{\Delta x}\right]. \tag{111}$$

If the material is assumed to be nonmagnetic, i.e., $\mu(i) = \mu_0$ then we can define a new constant, $R_b = \dfrac{\Delta t}{\mu \Delta x}$ and let $\widetilde{E} = R_b E$. Substituting this into the above equation yields,

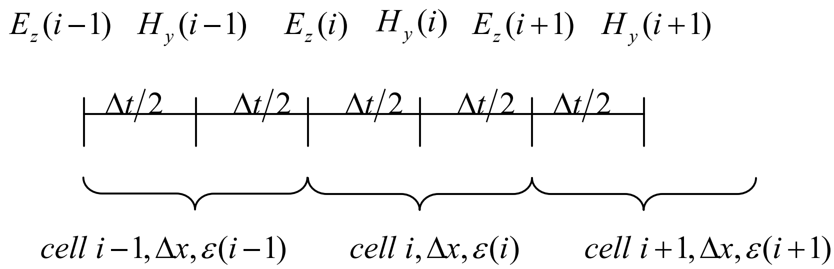

Figure 1: Computational lattice of a one-dimensional FDTD method.

$$H_y^{n+1/2}(i) = H_y^{n-1/2}(i) + \widetilde{E}_z^n(i) - \widetilde{E}_z^n(i+1). \tag{112}$$

This represents the discretization of the first equation. The second equation is derived in s similar fashion, from Ampere's law,

$$\frac{E_z^{n+1/2}(i) - E_z^{n-1/2}(i)}{\Delta t} = \frac{-1}{\in}\left[\frac{H_y^n(i) - H_y^n(i-1)}{\Delta x}\right] - \frac{\sigma}{\in} E_z^n(i). \tag{113}$$

At this point we will digress just a moment to explain the nature of the FDTD computational lattice. **Figure 1** illustrates 3 cells of a one-dimensional computational FDTD region. You may have wondered why the difference equation for E_z, Eq. (110) contains E_z (i+1) and E_z (i) whereas that for H_y, Eq. (113) contains H_y (i) and H_y (i-1). This has to do with the orientation of the fields in the computational cell. For instance, to perform the central difference on the E_z field component about H_y (i) we need to do the following

$$\left.\frac{\partial E_z}{\partial x}\right|_{about\ H_y(i)} = \frac{E_z^n(i+1) - E_z^n(i)}{\Delta x}, \tag{114}$$

according to **Figure 1**. Likewise to do the same derivative on the H_y field component about E_z (i), i.e., in the same computational cell, we do the following

$$\left.\frac{\partial H_y}{\partial x}\right|_{about\ E_z(i)} = \frac{H_y^n(i) - H_y^n(i-1)}{\Delta x}, \tag{115}$$

according to **Figure 1**. Therefore because the electric fields precede the magnetic fields within a computational cell, their difference equation uses the field from the next cell, whereas those for the magnetic field use the field component from the previous cell. Note also that a $\Delta t/2$ time step exists between each field component. This gives rise to the n – ½, n, and n + ½ notation for the difference equations in the time domain.

Now to continue with our derivation of the second equation, we can rearrange Eq. (113) into the following form

$$E_z^{n+1/2}(i) - E_z^{n-1/2}(i) + \frac{\sigma \Delta t}{\in} E_z^n(i) = -\frac{\Delta t}{\in}\left[\frac{H_y^n(i) - H_y^n(i-1)}{\Delta x}\right]. \tag{116}$$

However now we have three different time steps on the left hand side of our equation, $n - ½$, n, $n + ½$. To reduce this to just two time steps, i.e., a former and a present without and intermediate, and we can perform a linear interpolation on the E_z^n component

$$E_z^{n+1/2} = \frac{E_z^{n+1/2}(i) + E_z^{n-1/2}(i)}{2}. \tag{117}$$

Substituting this back in and combining like terms we get

$$E_z^{n+1/2}(i)\left(1 + \frac{\sigma \Delta t}{2 \in}\right) - E_z^{n-1/2}(i)\left(1 - \frac{\sigma \Delta t}{2 \in}\right) = -\frac{\Delta t}{\in}\left[\frac{H_y^n(i) - H_y^n(i-1)}{\Delta x}\right]. \tag{118}$$

Solving for the most recent time step yields

$$E_z^{n+1/2}(i) = E_z^{n-1/2}(i)\frac{\left(1 - \frac{\sigma \Delta t}{2 \in}\right)}{\left(1 + \frac{\sigma \Delta t}{2 \in}\right)} - \frac{\frac{\Delta t}{\in}}{\left(1 + \frac{\sigma \Delta t}{2 \in}\right)}\left[\frac{H_y^n(i) - H_y^n(i-1)}{\Delta x}\right]. \tag{119}$$

We can now define a new material dependent variable

$$C_a(i) = \frac{\left(1 - \frac{\sigma \Delta t}{2 \in}\right)}{\left(1 + \frac{\sigma \Delta t}{2 \in}\right)}. \tag{120}$$

Multiplying both sides of Eq. (119) by Eq. (120) and $R_b(i)$ we get

$$E_z^{n+1/2}(i) = E_z^{n-1/2}(i) C_a(i) - \frac{R_b(i)\frac{\Delta t}{\in}}{\left(1 + \frac{\sigma \Delta t}{2 \in}\right)}\left[\frac{H_y^n(i) - H_y^n(i-1)}{\Delta x}\right]. \tag{121}$$

Finally setting

$$C_b(i) = \frac{R_b(i)\frac{\Delta t}{\in}}{\Delta x\left(1 + \frac{\sigma \Delta t}{2 \in}\right)} \tag{122}$$

we get the final form of the second difference equation

$$\widetilde{E}_z^{n+1/2}(i) = \widetilde{E}_z^{n-1/2}(i) C_a(i) - C_b(i) = \left[H_y^n(i) - H_y^n(i-1)\right], \text{ or} \tag{123}$$

$$\widetilde{E}_z^{n+1/2}(i) = \widetilde{E}_z^{n-1/2}(i) C_a(i) + C_b(i) = \left[H_y^n(i-1) - H_y^n(i)\right] \tag{124}$$

Thus, Eqs. (450) and (455) represent the two main equations in the one-dimensional FDTD program. Next we consider the source.

2.2 1D FDTD Source

The incident wave, or source, for the computational region can be either a transient pulse or a sinusoid. For the case of a transient pulse we can determine the scattering response over the entire bandwidth of the pulse. However, in many applications a specific frequency is of interest. In this case we can use a sinusoidal source and allow for enough time steps to insure that the problem has reached a steady state condition. We will discuss the criterion for this later, but for now will simply introduce the mathematical forms for both sources.

2.2.1 Transient Pulse

To use a transient pulse as the incident wave we can assume $E_z\,(x,t=0)$ is a Gaussian pulse,

$$E_z(x) = E_0 \exp-\left(\beta(x - xp)^2\right), \tag{125}$$

where $\beta = -\ln(0.001)/(w\Delta x)^2$. Correspondingly, we can define $H_y(x,t=0)$ as

$$H_y(x) = \frac{E_0}{Z_0} \exp-\left(\beta\left(x - xp + \frac{\Delta x}{4}\right)^2\right). \tag{126}$$

The $\dfrac{\Delta x}{4}$ term comes form the spatial delay of defining $H_y^{n+1/2}(x)$ as it follows from $E_z(x)$. This can be seen from $H_y(x) = H_y\left(x + \dfrac{c\Delta t}{2}\right)$, where the $\dfrac{c\Delta t}{2}$ term results from the half time step, that results from going from E_2 to H_y. If we use a spatial sampling rate of $\Delta x = 2c\Delta t$, which as we will see later insures stability of the algorithm, then the argument of $H_y\left(x + \dfrac{c\Delta t}{2}\right) = H_y\left(x + \dfrac{c\Delta x}{2(2c)}\right) = H_y\left(x + \dfrac{\Delta x}{4}\right)$.

In general we can use any function as the initial condition, so long as it satisfies the homogeneous wave equation. This type of source is defined over all space as an initial condition. Once time marching begins then the source is not updated, rather it propagates through the computational medium according to Maxwell's equations.

2.2.2 Sinusoidal Pulse

For a sinusoidal pulse we define the electric field as a function of time, $E_z(x_i, t)$ at a given location x_i. This can be thought of as a radiating dipole in the sense that the field oscillates sinusoidally at that point for the duration of time marching. In this way the field is defined as

$$E_z(x_i) = E_0 \sin(2\pi f t) \tag{127}$$

where f is the frequency of the incident field. Note that in this formulation the field will travel to both the right and left, in contrast to the pulse. This is because we have not specified the magnetic field with a corresponding spatial delay. This can be done if a sinusoidal field that propagates in a single direction is desired. We next need to consider what happens at the end points of our 1D lattice. This is called an absorbing boundary and is discussed next.

2.2.3 Other Sources

There are also other types of sources that are more commonly used, such as the connecting boundary and the scattered field formulation, but we will save these for a later discussion.

2.3 1D Absorbing Boundary

At the truncation of the computational region we must apply an absorbing boundary otherwise the electric field will see an infinite impedance and be reflected back into the computational region, thereby giving rise to numerical errors. There are many types of absorbing boundaries, Radiation, Bayliss-Turkel, Enquist-Majda, Mur, Higdon, Liao, and Berenger. We will talk about each of these at great length later in the course, however, for now we will discuss the radiation ABC.

If we consider the wave equation

$$\frac{\partial^2 E_z}{\partial_x} - c^2 \frac{\partial^2 E_z}{\partial x^2} = 0 \tag{128}$$

which can be factored into the following expression

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} - \frac{1}{c}\frac{\partial}{\partial t}\right) E_z(x,t) = 0\,. \tag{129}$$

If we look at both differential operators we can see that the first one represents a traveling wave in the forward direction and the second one represents a traveling wave in the backward direction. This can be seen by considering that the solution to the wave equation is a combination of sinusoids of the form

$$E_z(x,t) = A\sin\left(\omega\left(t - x\sqrt{\mu \in_0}\right)\right) + B\sin\left(\omega\left(t + x\sqrt{\mu \in_0}\right)\right), \tag{130}$$

where ω is the angular frequency. If we apply $\frac{\partial}{\partial t} = 0$ operator to the argument of the sinusoid, we can determine the velocity of the propagating wave because this represents a constant point in time on phase of the wave, i.e., the relative motion on the wave as a function of time is zero. Doing this to the positive going wave we have:

$$\frac{\partial\left(\omega\left(t - x\sqrt{\mu \in_0}\right)\right)}{\partial t} = \omega - k\frac{\partial x}{\partial t} = \omega - k v_x = 0\text{, or} \tag{131}$$

$$v_x = \frac{\omega}{k}$$

which is a positive velocity in the x direction. As a result, we can write $E_z(x, t)$ as a function of forward and backward traveling waves

$$E_z(x,t) = Af\left(\omega\left(t - x\sqrt{\mu \in_0}\right)\right) + Bg\left(\omega\left(t + x\sqrt{\mu \in_0}\right)\right), \tag{132}$$

where f represents the forward wave and g represents the backward wave. Therefore at the last node of the computational space, N, we must satisfy the positive going wave condition,

$$\left(\frac{\partial}{\partial x} + \frac{1}{c}\frac{\partial}{\partial t}\right) E_z^+(x,t) = 0 \tag{133}$$

In difference form this equates to

$$\frac{E_z^n(N) - E_z^n(N-1)}{\Delta x} + \frac{1}{c}\frac{E_z^{n+1/2}(N-1/2) - E_z^{n-1/2}(N-1/2)}{\Delta t} = 0. \tag{134}$$

By interpolating the half time-step components we get

$$E_z^{n+1/2}(N) = \frac{E_z^n(N) + E_z^{n+1}(N)}{2} \tag{135}$$

and also interpolating the half space-step we get

$$E_z^{n+1/2}(N-1/2) = \frac{E_z^{n+1/2}(N) - E_z^{n+1/2}(N-1)}{2} \tag{136}$$

Doing the same for $E_z^{n-1/2}(N-1/2)$ term and substituting back into Eq. (376) we get the following

$$E_z^n(N) = E_z^{n-1}(N-1) + \left(\frac{c\Delta t - \Delta x}{c\Delta t + \Delta x}\right)\left[E_z^n(N-1) - E_z^{n-1}(N)\right]. \tag{137}$$

If we use the magic time step as the stability criterion, $\Delta t = \Delta x / c$ then we have the following radiation condition

$$E_z^n(N) = E_z^{n-1}(N-1). \tag{138}$$

Note that the n-1 actually corresponds to a two time-step delay due to the interlaced computation of the electric and magnetic field. For the backward going wave we can derive a similar relationship,

$$E_z^n(1) = E_z^{n-1}(2). \tag{139}$$

Next we need to discuss two more critical aspects of the FDTD method in order to formulate and apply it to a problem they are: numerical dispersion and stability.

2.4 Numerical Dispersion

When implementing the FDTD method it is necessary to establish the conditions under which stability of the technique can be insured. For the FDTD method this implies that as time marching continues, according to the Yee algorithm, the electric and magnetic field values do not grow without bound. Essentially dispersion, including numerical dispersion, is a relation between the wave number and the frequency. To examine this consider the wave equation in one-dimension:

$$\frac{\partial^2 U}{\partial t^2} - c^2 \frac{\partial^2 U}{\partial x^2} = 0\,, \tag{140}$$

which has the solution of the form

$$U(x,t) = e^{j(\omega t - kx)}\,. \tag{141}$$

If we substitute this solution into Eq. (140) we get the following:

$$\begin{aligned} (j\omega)^2 e^{j(\omega t - kx)} &= c^2(-jk)^2 e^{j(\omega t - kx)} \\ \omega^2 = c^2 k^2\,,\ \ &or\ \omega = ck \end{aligned} \tag{142}$$

Thus, the wave number is linearly proportional to the frequency, with the speed of light being the constant of proportionality. This may be more widely known as the phase velocity relationship when expressed as $v_p = \omega / k$ where $v_p = c$ for free space.

To determine how the FDTD effects this relation we repeat the same procedure using difference equations. Note that we can represent the second order difference equation (for both space and time) in terms of,

$$\left.\frac{\partial^2 U}{\partial x^2}\right|_{x_i, t_n} \cong \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2}. \tag{143}$$

Substituting these expressions into the wave equation and solving for the most recent time value, we get,

$$U_i^{n+1} = (c\Delta t)^2 \left(\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2} \right) + 2U_i^n - U_i^{n-1}. \tag{144}$$

Now defining the difference form of Eq. (141)

$$U_i^n = e^{j\left(\omega n\Delta t - \tilde{k} i \Delta x\right)}, \tag{145}$$

where $\tilde{k}$ is the wave number of a sinusoidally traveling wave frequency of ω, which exists in the FDTD solution space due to discretization. After substituting Eq. (145) in Eq. (144) we get:

$$\begin{aligned} e^{j\left(\omega(n+1)\Delta t - \tilde{k} i \Delta x\right)} &= \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(e^{j\left(\omega n\Delta t - \tilde{k}(i+1)\Delta x\right)} - 2e^{j\left(\omega n\Delta t - \tilde{k} i \Delta x\right)} + e^{j\left(\omega n\Delta t - \tilde{k}(i-1)\Delta x\right)} \right) + \\ &\left(2e^{j\left(\omega n\Delta t - \tilde{k} i \Delta x\right)} - e^{j\left(\omega(n-1)\Delta t - \tilde{k} i \Delta t\right)} \right) \end{aligned}. \tag{146}$$

Factoring out the $e^{j\left(\omega n\Delta t - \tilde{k} i \Delta x\right)}$ yields

$$e^{j\omega\Delta t} = \left(\frac{c\Delta t}{\Delta x}\right)^2 \left(e^{-j\tilde{k}\Delta x} - 2 + e^{j\tilde{k}\Delta x} \right) + \left(2 - e^{-j\omega\Delta t} \right). \tag{147}$$

If we group the time and spatial terms and divide both sides by 2 we get:

$$\frac{e^{j\omega\Delta t}-e^{-j\omega\Delta t}}{2}=\left(\frac{c\Delta t}{\Delta x}\right)^2\left(\frac{e^{-j\tilde{k}\Delta x}+e^{j\tilde{k}\Delta x}}{2}-1\right)+1, \tag{148}$$

which according to Euler's identity yields,

$$\cos(\omega\Delta t)=\left(\frac{c\Delta t}{\Delta x}\right)^2\left[\cos\left(\tilde{k}\Delta x\right)-1\right]+1. \tag{149}$$

This is a nonlinear equation that represents the relationship between the wave number and frequency, which implies that the phase velocity is not linearly proportional to the frequency. This means that as a simulated wave propagates through the solution space it undergoes phase errors, because the numerical wave either slows or accelerates relative to the actual wave propagation in a physical space. To examine the implications of Eq. (149) we consider three special cases below.

2.4.1 Case 1

The first case that we consider is when Δt→0 and Δx→0. In this case we use the first two terms of the Taylor series expansion in Eq. (149)

$$1-\frac{(\omega\Delta t)^2}{2}\left[1-\left(\tilde{k}\Delta x\right)^2-1\right]+1$$

$$\frac{(\omega\Delta t)^2}{2}=\left(\frac{c\Delta t}{\Delta x}\right)^2\left(\tilde{k}\Delta x\right)^2$$

$$\tilde{k}=\pm\frac{\omega}{c}$$

which is equal to the analytic case. This implies that as we reduce the size of the spatial and time increments we actually approach the analytic solution. However, this comes at the expense of increasing the computational and memory requirements. Therefore it is necessary to see the bounds on the spatial and time increments, which is examined in the next two cases.

2.4.2 Case 2

In this case we consider the relation $c\Delta t = \Delta x$. This is often referred to as the magic time step. If we substitute this into Eq.(149) we get,

$$\cos(\omega\Delta t)=\left(\frac{c\Delta t}{\Delta x}\right)^2\left[\cos\left(\tilde{k}\Delta x\right)-1\right]+1, \tag{150}$$

$$\cos(\omega\Delta t)=\cos\left(\tilde{k}\Delta x\right), \tag{151}$$

which can only be true if the two arguments of the cosine terms are equal. That is to say $\omega\Delta t=\tilde{k}\Delta x$ or $\tilde{k}=\frac{\omega}{c}$. This is quite remarkable in that if this spatial and time relation is used there is absolutely no numerical dispersion. However, as we will see in the next section this has great implications in stability, which unfortunately limits its application. The next case we consider is the general case where we solve the non-linear equation in terms of the wave number.

2.4.3 Case 3

In this case we consider the general case were there is no assumed relation between Δt and Δx. Therefore in this approach we need to solve for a general expression that can tell us the effect of numerical dispersion for a given relationship. To do this we simple solve Eq. (149) for the numerical wave number

$$\tilde{k} = \frac{1}{\Delta x}\cos^{-1}\left\{1 + \left(\frac{\Delta x}{c\Delta t}\right)^2 \left[\cos(\omega\Delta t) - 1\right]\right\}. \qquad (152)$$

Using this equation we can determine the numerical wave number for any relation between Δt and Δx for a given sampling, i.e., $\lambda/10$. That is we simply substitute in the appropriate parameters and determine the effect of discretizing our space. For example if we use the relation $c\Delta t = \Delta x/2$ and $\Delta x = \lambda/10$, then

$$\tilde{k} = \frac{1}{\Delta x}\cos^{-1}\left\{1 + 4\left[\cos\left(\frac{k\Delta x}{2}\right) - 1\right]\right\}, \qquad (153)$$

where I have used the analytic relation $ck = \omega$. Using the sampling rate of $\Delta x = \lambda/10$ we get,

$$\tilde{k} = \frac{1}{\Delta x}\cos^{-1}(0.08042) = \frac{0.6364}{\Delta x}. \qquad (154)$$

Now defining numerical phase velocity as $\tilde{v}_p = \frac{\omega}{\tilde{k}}$, we get $\tilde{v}_p = 0.9873c$. Thus over a distance of 10λ, i.e., 100 cells, the numerical wave would only propagate 98.73 cells, which gives a phase error of 45.72°. This is a substantial error, more than $\pi/4$. This is an indication that $\Delta x = \lambda/10$ is an insufficient sampling rate. If we consider $\Delta x = \lambda/20$ and every thing else the same, then $\tilde{v}_p = 0.9968c$ with a corresponding phase error at a distance of 10λ of 11.19°. The conclusion to draw from this is that the larger the solution space the greater the phase error that results from numerical dispersion. Therefore in the application of scattering analysis it is imperative to reduce the solution region to the smallest possible. Otherwise as the fields propagate they undergo dispersion and interfere incorrectly, thereby producing incorrect results.