EE840 COMPUTATIONAL ELECTROMAGNETICS – Fall 2002

Lecture #6: Introduction to Higher Dimensional FDTD

1 Key points from last lecture:

1. Completed the introducing of the 1D FDTD method.
2. Discussed the implementation of a Gaussian and hard sinusoidal source.
3. Presented the ideal radiation boundary condition.
4. Over viewed the layout of a 1D FDTD program.

2 Today's Objectives

1. Complete discussion of stability for 1D FDTD.
2. Introduce the 2D and 3D FDTD formulations.
3. Discuss mesh material properties.

2.1 Stability

As stated before, the FDTD method approximates Maxwell's equations as a set of coupled difference equations. As such the method is useful only when the solution of the difference equations are convergent and stable. Convergence means that, as time stepping continues the solution to the difference equations asymptotically approaches the solution given by Maxwell's equations. On the other hand stability is basically stated as a set of conditions under which the error generated by the finite difference approximation is finite and does not grow in an unbounded fashion as time stepping progresses. As a result there is a mathematical relation to describe this problem.

To see consider the requirement for Δt in the difference equations. As shown before, the solution for a steady state field in a lossless medium is of the form:

$$f(x, y, z, t) = f_0 e^{j\omega t} . \tag{155}$$

This function can also be thought of as the solution to the following differential equation

$$\frac{\partial f}{\partial t} = j\omega f . \tag{156}$$

If we approximate this equation in the difference equation, we get the following

$$\frac{f^{n+\frac{1}{2}} - f^{n-\frac{1}{2}}}{\Delta t} = +j\omega f^{n} . \tag{157}$$

When Δt is small, we can define a numerical increment factor q

$$q = \frac{f^{n+\frac{1}{2}}}{f^n} = \frac{f^n}{f^{n-\frac{1}{2}}}, \tag{158}$$

which tells us the time evolution of the fields at a give point in space. Thus, if for subsequent time steps this expression increases, then $|q| > 1$ and the field values grow without bound and are therefore considered unstable. Therefore we need to establish the conditions under which $|q| < 1$. To do this substitute Eq. (158) into Eq. (157) and solve for the following quadratic equation,

$$q^2 - j\omega\Delta t q - 1 = 0 \tag{159}$$

The solution to this equation is

$$q = \frac{+ j\omega\Delta t}{2} \pm \sqrt{1 - \left(\frac{\omega\Delta t}{2}\right)^2} . \tag{160}$$

Numerical stability requires that as the number of time steps approaches infinity, i.e., as n →∞, and given Δt is small enough, that the numerical increment factor $|q| \le 1$. According to Eq. (160) the condition sufficient for $|q| \le 1$ is $\frac{\omega\Delta t}{2} < 1$ or $\Delta t < \frac{2}{\omega}$. Since $\omega = 2\pi/T$ we have

$$\Delta t < \frac{T}{\pi}. \tag{161}$$

Next consider the time step requirement in relation to the spatial step Δx for a given steady state field. In a Cartesian coordinate system, Maxwell's equations result to the homogenous scalar wave equation:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\omega^2}{c^2} f = 0 \tag{162}$$

As shown before the solution to this equation is a plane wave of the form

$$f(x,t) = f_0 e^{j(\omega t - kx)} \tag{163}$$

Substituting this expression into a difference form of Eq. (162), i.e., using the second order difference expression,

$$\frac{\partial^2 f}{\partial x^2} \approx \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{(\Delta x)^2}, \tag{164}$$

we get

$$\frac{\partial^2 f}{\partial x^2} \approx \frac{e^{jk\Delta x} - 2 + e^{-jk\Delta x}}{(\Delta x)^2} f = -\frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)^2} f \tag{165}$$

and the result is,

$$\frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)^2} - \frac{\omega^2}{c^2} = 0 \tag{166}$$

This equation gives us the relation between the frequency ω and the wave vector k for a plane wave for the discretized form of the wave equation. If we combine this with Eq. (161) we have

$$\left(\frac{c\Delta t}{2}\right)^2 \frac{\sin^2\left(\frac{k\Delta x}{2}\right)}{\left(\frac{\Delta x}{2}\right)^2} = \left(\frac{\omega\Delta t}{2}\right)^2 \leq 1 \tag{167}$$

Since $\sin^2(\theta)$ is bounded by 1, the following condition must be met,

$$(c\Delta t)^2 \frac{1}{(\Delta x)^2} \leq 1 \tag{168}$$

in order for the above relation to be satisfied for any wave number, k. This reduces tot he simple stability condition,

$$c\Delta t \leq \Delta x\,. \tag{169}$$

This is the condition under which the difference solution will be stable. For higher dimensions we have the form $c\Delta t \leq \Delta x/\sqrt{D}$, where D = 1, 2, or 3, depending on the number of dimensions used in the solution space. This will be discussed later, for now we will focus on how to implement all of the above ideas ina single program.

2.2 Writing a 1D FDTD Program

1. Define physical constants: μ, $\in$, (x), Z_0, Ca, and Cb.

2. Define program constants: E_0, Δt, Δx, N_t (total number of time steps), and N_x (total number of nodes).

3. If using s pulse source, define E_z Nd H_y, otherwise implement hard sources for E_z.

4. Start the time marching:

   - pulse: $n = 1 : N_t$
   - sinusoid: $n = 1 : N_t$

5. Increment spatial index:
   if $i = 1$
   $E_z new(1) = E_z old(2)$

   else if $i = Nx$
   $E_z new(N) = E_z old(N - 1)$

   else
   $E_z new(i) = Ca(i) E_z old(i) + Cb(i) \left[H_y old(i - 1) - H_y old(i) \right]$

   end

   separate loop
   $H_y new(i) = H_y old(i) - E_z new(i + 1) = E_z new(i)$

1. Store previous time variables
   $H_y old = H_y new$
   $E_z old = E_z new$

Go To Step 4

3 General 3D Formulation

So far we have considered a simple 1D FDTD formulation, which physically represents the propagation of an infinite plane wave. More often than not, electromagnetic boundary value problems cannot be accurately represented as a 1D problem. In these cases it becomes necessary to use a more general formulation of the FDTD method. To this end we will discuss the 3D formulation in this section.

Recall that Maxwell's curl equations are:

$$\begin{aligned} \nabla \times E &= -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t} \\ \nabla \times H &= \frac{\partial D}{\partial t} + J = \varepsilon \frac{\partial E}{\partial t} + \sigma E. \end{aligned} \tag{170}$$

For simplicity we will only consider their solution for a linear and isotropic medium. In Cartesian coordinates, the curl equations can be written as:

$$\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\mu \frac{\partial H_x}{\partial t} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\mu \frac{\partial H_y}{\partial t} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\mu \frac{\partial H_z}{\partial t} \end{aligned} \tag{171}$$

and

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = -\mu \frac{\partial E_x}{\partial t} + \sigma E_x$$
$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = -\mu \frac{\partial E_y}{\partial t} + \sigma E_y \tag{172}$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = -\mu \frac{\partial E_z}{\partial t} + \sigma E_z .$$

As discussed before we can this set of coupled differential equations as a set of coupled difference equations, which is discussed next.

3.1 3D Discretization

Recall that a function $f(\boldsymbol{x, y, z, t})$ can be represented in discrete form as:

$$f(x, y, z, t) = f(i\Delta z, j\Delta y, k\Delta z, n\Delta t) = f^n(i, j, k). \tag{173}$$

In general, the FDTD space is sampled in a uniform fashion in which case we can represent the spatial step size as

$$\Delta x = \Delta y = \Delta z = \delta . \tag{174}$$

Using this approach we can represent the electric and magnetic field at the individual nodes (*i, j, k*). Note, that the *E* and *H* nodes are only defined where (*i, j, k*) are integers. Because of this the field values at the points in-between the nodes are undefined!

According to the difference approximation, the spatial and temporal derivatives are approximated as,

$$\frac{\partial f(x, y, z, t)}{\partial x} \approx \frac{f^n(i + \frac{1}{2}, j, k) - f^n(i - \frac{1}{2}, j, k)}{\delta}$$
$$\frac{\partial f(x, y, z, t)}{\partial y} \approx \frac{f^n(i, j + \frac{1}{2}, k) - f^n(i, j - \frac{1}{2}, k)}{\delta} \tag{175}$$
$$\frac{\partial f(x, y, z, t)}{\partial z} \approx \frac{f^n(i, j, k + \frac{1}{2}) - f^n(i, j, k - \frac{1}{2})}{\delta}$$

and

$$\frac{\partial f(x, y, z, t)}{\partial x} \approx \frac{f^{n+\frac{1}{2}}(x, y, z) - f^{n-\frac{1}{2}}(x, y, z)}{\delta} \tag{176}$$

3.2 FDTD Discretization of 3-D Curl Equations

Using the above difference expression in the first of Maxwell's curl equations and assuming that the point (*x, y, z, t*) is an H_x node, (*i, j* + ½. *k* + ½, *n*) we have:

$$\frac{H_x^{n+\frac{1}{2}}(i, j+\frac{1}{2}, k+\frac{1}{2}) - H_x^{n-\frac{1}{2}}(i, j+\frac{1}{2}, k+\frac{1}{2})}{\Delta t} = \frac{1}{\mu_0 \mu_r(i, j+\frac{1}{2}, k+\frac{1}{2})} \cdot \left[\frac{E_y^n(i, j+\frac{1}{2}, k+1) - E_y^n(i, j+\frac{1}{2}, k)}{\Delta z} - \frac{E_y^n(i, j+1, k+\frac{1}{2}) - E_y^n(i, j, k+\frac{1}{2})}{\Delta y}\right] \tag{177}$$

Similarly, the other two equations resulting from Eq. (171) can be written as:

$$\frac{H_x^{n+\frac{1}{2}}(i+\frac{1}{2}, j, k+\frac{1}{2}) - H_x^{n-\frac{1}{2}}(i+\frac{1}{2}, j, k+\frac{1}{2})}{\Delta t} = \frac{1}{\mu_0 \mu_r(i+\frac{1}{2}, j, k+\frac{1}{2})} \left[\frac{E_y^n(i+1, j, k+\frac{1}{2}) - E_y^n(i, j, k+\frac{1}{2})}{\Delta x} - \frac{E_y^n(i+\frac{1}{2}, j, k+1) - E_y^n(i+\frac{1}{2}, j, k)}{\Delta z}\right] \text{ and} \tag{178}$$

$$\frac{H_x^{n+\frac{1}{2}}(i+\frac{1}{2}, j+\frac{1}{2}, k) - H_x^{n-\frac{1}{2}}(i+\frac{1}{2}, j+\frac{1}{2}, k)}{\Delta t} = \frac{1}{\mu_0 \mu_r(i+\frac{1}{2}, j+\frac{1}{2}, k)} \left[\frac{E_y^n(i+\frac{1}{2}, j+1, k) - E_y^n(i+\frac{1}{2}, j, k)}{\Delta y} - \frac{E_y^n(i+1, j+\frac{1}{2}, k) - E_y^n(i, j+\frac{1}{2}, k)}{\Delta x}\right]. \tag{179}$$

In a corresponding fashion the first relation in Eq. (172) can be represented according to the following difference equation:

$$\frac{E_x^{n+1}(i+\frac{1}{2}, j, k) - E_x^n(i+\frac{1}{2}, j, k)}{\Delta t} = \frac{1}{\varepsilon_0 \mu_r(i+\frac{1}{2}, j, k)} \cdot \left[\frac{H_y^{n+\frac{1}{2}}(i+\frac{1}{2}, j+\frac{1}{2}, k) - H_y^{n+\frac{1}{2}}(i+\frac{1}{2}, j-\frac{1}{2}, k)}{\Delta y} - \frac{H_y^{n+\frac{1}{2}}(i+\frac{1}{2}, j, k+\frac{1}{2}) - H_y^{n+\frac{1}{2}}(i+\frac{1}{2}, j, k-\frac{1}{2})}{\Delta z} - \sigma(i+\frac{1}{2}, j, k)\frac{E_x^{n+1}(i+\frac{1}{2}, j, k) + E_x^n(i+\frac{1}{2}, j, k)}{2}\right] \tag{180}$$

where we have used the following interpolation,

$$E_x^{n+\frac{1}{2}}(i+\frac{1}{2}, j, k) = \frac{E_x^{n+1}(i+\frac{1}{2}, j, k) + E_x^n(i+\frac{1}{2}, j, k)}{2}. \tag{181}$$

Similarly, the other two equations can be represented as:

$$\frac{E_x^{n+1}(i,j+\frac{1}{2},k)-E_x^{n}(i,j+\frac{1}{2},k)}{\Delta t}=\frac{1}{\varepsilon_0\varepsilon_r(i,j+\frac{1}{2},k)}\cdot$$

$$\left[\frac{H_y^{n+\frac{1}{2}}(i,j+\frac{1}{2},k+\frac{1}{2})-H_y^{n+\frac{1}{2}}(i,j+\frac{1}{2},k-\frac{1}{2})}{\Delta z}-\frac{H_y^{n+\frac{1}{2}}(i+\frac{1}{2},j+\frac{1}{2},k)-H_y^{n+\frac{1}{2}}(i-\frac{1}{2},j+\frac{1}{2},k)}{\Delta x}-\right.$$

$$\left.\sigma(i,j+\frac{1}{2},k)\frac{E_x^{n+1}(i,j+\frac{1}{2},k)+E_x^{n}(i,j+\frac{1}{2},k)}{2}\right]$$

(182)

$$\frac{E_x^{n+1}(i,j,k+\frac{1}{2})-E_x^{n}(i,j,k+\frac{1}{2})}{\Delta t}=\frac{1}{\varepsilon_0\varepsilon_r(i,j,k+\frac{1}{2})}\cdot$$

$$\left[\frac{H_y^{n+\frac{1}{2}}(i+\frac{1}{2},j,k+\frac{1}{2})-H_y^{n+\frac{1}{2}}(i-\frac{1}{2},j,k+\frac{1}{2})}{\Delta x}-\frac{H_y^{n+\frac{1}{2}}(i,j+\frac{1}{2},k+\frac{1}{2})-H_y^{n+\frac{1}{2}}(i,j-\frac{1}{2},k+\frac{1}{2})}{\Delta y}-\right.$$

$$\left.\sigma(i,j,k+\frac{1}{2})\frac{E_x^{n+1}(i,j,k+\frac{1}{2})+E_x^{n}(i,j,k+\frac{1}{2})}{2}\right].$$

(183)

The above difference equations can be simplified if we set $\mu_r = 1$, use the relation $\Delta x = \Delta y = \Delta z = \delta$, and define the following material constants:

$$R=\frac{\Delta t}{2\varepsilon_0},R_a=\frac{(\Delta t)^2}{\delta\mu_0\varepsilon_0},R_b=\frac{\Delta t}{\mu_0\delta}$$
$$Ca(m)=\frac{1-R\delta(m)/\varepsilon_r(m)}{1+R\delta(m)/\varepsilon_r(m)} \tag{184}$$
$$Cb(m)=\frac{R_a}{\varepsilon_r(m)+R\delta(m)}.$$

The index m in the above equation stands for a set of discretized coordinates, for example in Eq. (183) m is equal to (i + ½, j, k), (i, j + ½, k), and (i, j, k + ½) for each equation. Substituting these parameters back into the difference equations yields:

$$\begin{aligned}&H_x^{n+\frac{}{2}}(i,j+\frac{1}{2},k+\frac{1}{2})=H_z^{n-\frac{1}{2}}(i,j+\frac{1}{2},k+\frac{1}{2})+\widetilde{E}_y^{n}(i,j+\frac{1}{2},k+1)-\\&\widetilde{E}_y^{n}(i,j+\frac{1}{2},k)+\widetilde{E}_z^{n}(i,j,k+\frac{1}{2})-\widetilde{E}_z^{n}(i,j+1,k+\frac{1}{2})\end{aligned}, \tag{185}$$

$$\begin{aligned}&H_z^{n+\frac{}{2}}(i+\frac{1}{2},j,k+\frac{1}{2})=H_z^{n-\frac{1}{2}}(i+\frac{1}{2},j,k+\frac{1}{2})+\widetilde{E}_x^{n}(i+1,j,k+\frac{1}{2})-\\&\widetilde{E}_z^{n}(i,j,k+\frac{1}{2})+\widetilde{E}_z^{n}(i+\frac{1}{2},j,k)-\widetilde{E}_x^{n}(i+\frac{1}{2},j,k+1)\end{aligned}, \tag{186}$$

$$H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2},j+\tfrac{1}{2},k)=H_z^{n-\frac{1}{2}}(i+\tfrac{1}{2},j+\tfrac{1}{2},k)+\widetilde{E}_x^n(i+\tfrac{1}{2},j+1,k)- \\ \widetilde{E}_z^n(i+\tfrac{1}{2},j,k)+\widetilde{E}_z^n(i,j+\tfrac{1}{2},k)-\widetilde{E}_x^n(i+1,j+\tfrac{1}{2},k) \quad , \tag{187}$$

$$\widetilde{E}_x^{n+1}(i+\tfrac{1}{2},j,k)=Ca(m)\widetilde{E}_z^n(i+\tfrac{1}{2},j,k)+Cb(m)\Big[H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2},j+\tfrac{1}{2},k) \\ -H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2},j-\tfrac{1}{2},k)+H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2},j,k-\tfrac{1}{2})-H_x^{n+\frac{1}{2}}(i+\tfrac{1}{2}j,k+\tfrac{1}{2})\Big] \quad , \tag{188}$$

$$\widetilde{E}_x^{n+1}(i,j+\tfrac{1}{2},k)=Ca(m)\widetilde{E}_z^n(i,j+\tfrac{1}{2},k)+Cb(m)\Big[H_y^{n+\frac{1}{2}}(i,j+\tfrac{1}{2},k+\tfrac{1}{2}) \\ -H_y^{n+\frac{1}{2}}(i,j+\tfrac{1}{2},k-\tfrac{1}{2})+H_y^{n+\frac{1}{2}}(i-\tfrac{1}{2},j+\tfrac{1}{2},k)-H^{n+\frac{1}{2}}(i+\tfrac{1}{2}j+\tfrac{1}{2},k)\Big] \quad , \tag{189}$$

$$\widetilde{E}_x^{n+1}(i,j,k+\tfrac{1}{2})=Ca(m)\widetilde{E}_z^n(i,j,k+\tfrac{1}{2})+Cb(m)\Big[H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2},j,k+\tfrac{1}{2}) \\ -H_y^{n+\frac{1}{2}}(i-\tfrac{1}{2},j,k-\tfrac{1}{2})+H_y^{n+\frac{1}{2}}(i,j-\tfrac{1}{2},k+\tfrac{1}{2})-H_x^{n+\frac{1}{2}}(i,j+\tfrac{1}{2},k+\tfrac{1}{2})\Big]. \tag{190}$$

Now let's see try to better understand the physical meaning of Eqs. (360) through (365). If we refer to **Figure 1**, we can see that the H_x, located in the center of the back face of cube, is dependent on the spatial derivatives of the surrounding E field components, i.e., the curl of E_y and E_x. Also according to Eq. (360) it depends on it previous value, which is located in the center of the front face of the cube. We should also note that in between the H_x values is a ring of three other field components, namely H_y, H_z, and E_x. This implies that for each time step a set of electric and magnetic fields are used to generate a set of electric and magnetic field components at the next time step. This leap frog process provides a tremendous amount of insight in to the physics of propagation.

The following table illustrates the leap frog process for the electric and magnetic fields at various nodes with their corresponding time and spatial coordinates.

Therefore according to Eqs. (360) through (365), we can formulate an iterative algorithm for the computation of electric and magnetic field components. This was the major contribution of Kane Yee in 1966.

Now let's try to understand the meaning of the index m. In the problem of wave propagation and scattering, the scattering object can be divided into several parts, each of which contains a uniform medium, as shown in **Figure 3**. As a result m can be thought of as a transformation variable that maps a three dimensional material coefficient in to a one-dimensional coefficient for ease of computation. Therefore in the case of **Figure 3** the whole space has only 3 different values: conductor, medium, and vacuum.

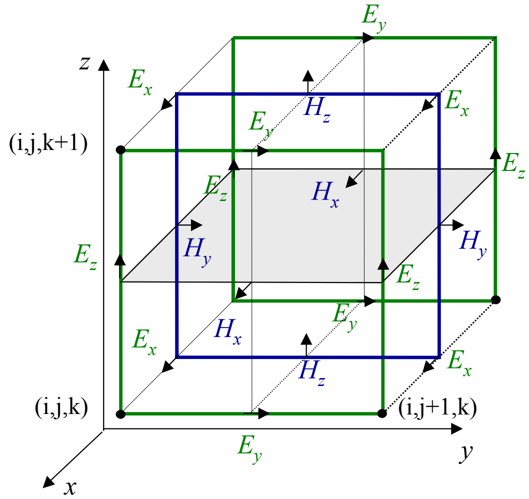

Figure 1: Three-dimensional FDTD lattice, note that the gray region applies to the two-dimensional lattice.

		x	y	z	t
E node	E_x	$i+\frac{1}{2}$	j	k	n
	E_y	i	$j+\frac{1}{2}$	k	n
	E_z	i	j	$k+\frac{1}{2}$	n
H node	H_x	i	$j+\frac{1}{2}$	$k+\frac{1}{2}$	$n\pm\frac{1}{2}$
	H_y	$i+\frac{1}{2}$	j	$k+\frac{1}{2}$	$n\pm\frac{1}{2}$
	H_z	$i+\frac{1}{2}$	$j+\frac{1}{2}$	k	$n\pm\frac{1}{2}$

Figure 2: Illustration of the spatial and temporal relation of the *E* and *H* nodes in a 3D FDTD computational lattice.

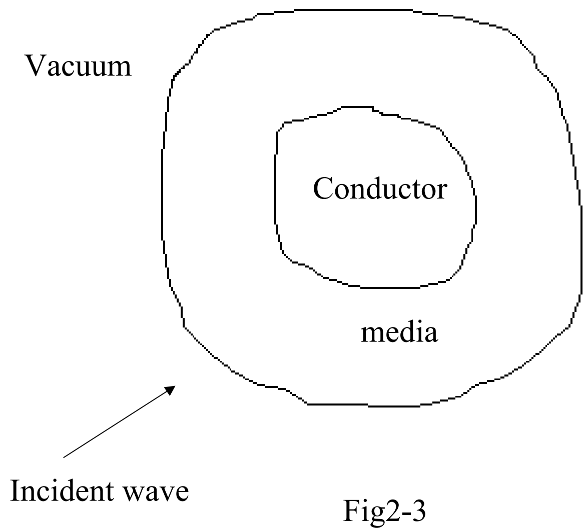

Figure 3: Illustration of a homogeneous material medium for the 2D FDTD.

4 2D FDTD Formulation

In the formulation of the two-dimensional FDTD method one can assume either TE-polarization (electric field perpendicular to the plane of incidence) or TM-polarization (magnetic field perpendicular to the plane of incidence). This results from the fact that in two-dimensions neither the fields nor the object contain any variations in the *z*-direction, consequently, all of the partial derivatives with respect to *z* are zero. Therefore, Maxwell's equations, in a rectangular coordinate system, can be reduced to two decoupled sets of equations,

$$\begin{aligned}\frac{\partial E_z}{\partial t} &= \frac{1}{\in}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \sigma E_z\right)\\ \frac{\partial H_x}{\partial t} &= \frac{1}{\mu}\frac{\partial E_z}{\partial y}\\ \frac{\partial H_y}{\partial t} &= \frac{1}{\mu}\frac{\partial E_z}{\partial x}\end{aligned} \tag{191}$$

and

$$\frac{\partial E_x}{\partial t} = \frac{1}{\in}\frac{\partial H_z}{\partial y}$$
$$\frac{\partial E_y}{\partial t} = \frac{1}{\in}\frac{\partial H_z}{\partial x} \qquad (192)$$
$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right).$$

The first set includes H_x and H_y as components of the magnetic field in the x and y direction, respectively, and the E_z as the component of the electric field in the z direction. The second set, includes E_x and E_y as components of the electric field in the x and y direction, respectively, and H_z as the component of magnetic field in z direction. Applying the central difference expressions, as presented before, the equations for the TM mode are derived to be the following:

$$H_x^{n+\frac{1}{2}}(i, j+\tfrac{1}{2}) = H_x^{n-\frac{1}{2}}(i, j+\tfrac{1}{2}) + \widetilde{E}_x^{\,n}(i, j) - \widetilde{E}_x^{\,n}(i, j+1)$$
$$H_x^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j) = H_x^{n-\frac{1}{2}}(i+\tfrac{1}{2}, j) + \widetilde{E}_x^{\,n}(i+1, j) - \widetilde{E}_x^{\,n}(i, j) \qquad (193)$$
$$\widetilde{E}_z^{\,n+!}(i, j) = Ca(m)\widetilde{E}_z^{\,n}(i, j) + Cb(m)\Big[H_y^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j) -$$
$$H_y^{n+\frac{1}{2}}(i-\tfrac{1}{2}, j) + H_x^{n+\frac{1}{2}}(i, j-\tfrac{1}{2}) - H_x^{n+\frac{1}{2}}(i, j+\tfrac{1}{2})\Big]$$

Similarly the TE case can be written as:

$$H_x^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j+\tfrac{1}{2}) = H_z^{n-\frac{1}{2}}(i+\tfrac{1}{2}, j+\tfrac{1}{2}) + \widetilde{E}_x^{\,n}(i+\tfrac{1}{2}, j+1)$$
$$-\widetilde{E}_x^{\,n}(i+\tfrac{1}{2}, j) + \widetilde{E}_y^{\,n}(i, j+\tfrac{1}{2}) - \widetilde{E}_y^{\,n}(i+1, j+\tfrac{1}{2})$$
$$\widetilde{E}_x^{\,n+1}(i+\tfrac{1}{2}, j) = Ca(m)\widetilde{E}_x^{\,n}(i+\tfrac{1}{2}, j) + Cb(m)\Big[H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j+\tfrac{1}{2})$$
$$-H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j-\tfrac{1}{2})\Big] \qquad (194)$$
$$\widetilde{E}_x^{\,n+1}(i, j+\tfrac{1}{2}) = Ca(m)\widetilde{E}_x^{\,n}(i, j+\tfrac{1}{2}) + Cb(m)\Big[H_z^{n+\frac{1}{2}}(i-\tfrac{1}{2}, j+\tfrac{1}{2})$$
$$-H_z^{n+\frac{1}{2}}(i+\tfrac{1}{2}, j+\tfrac{1}{2})\Big]$$

In Eq. (199) we shift the spatial index corresponding to (x, y) by ½, then coordinates (x, y) shift by $\delta/2$ along x and y axis. Also shifting the time step t, by $\Delta t/2$ Eq. (199) changes to:

$$\widetilde{E}_x^{\,n+\frac{1}{2}}(i, j+\tfrac{1}{2}) = Ca(m)\widetilde{E}_x^{\,n-\frac{1}{2}}(i, j+\tfrac{1}{2}) + Cb(m)\Big[H_z^{n+\frac{1}{2}}(i, j+1) - H_z^{n}(i, j)\Big]$$
$$\widetilde{E}_x^{\,n+\frac{1}{2}}(i+\tfrac{1}{2}, j) = Ca(m)\widetilde{E}_x^{\,n-\frac{1}{2}}(i+\tfrac{1}{2}, j) + Cb(m)\Big[H_z^{n+\frac{1}{2}}(i, j) - H_z^{n}(i+1, j)\Big]$$
$$H_z^{n+1}(i, j) = H_z^{n}(i, j) + \widetilde{E}_x^{\,n+\frac{1}{2}}(i, j+\tfrac{1}{2}) - \widetilde{E}_x^{\,n+\frac{1}{2}}(i, j-\tfrac{1}{2}) + \widetilde{E}_y^{\,n+\frac{1}{2}}(i-\tfrac{1}{2}, j) - \widetilde{E}_y^{\,n+\frac{1}{2}}(i+\tfrac{1}{2}, j)$$
(195)

In this way the *E,H* nodes in a 2D array, corresponding to Eqs. (338) and (200) can be shown as in **Figure 3** in the next section.
The application of the FDTD method to the solution of 2D problems can be succinctly summed up as an iterative algorithm, as shown in **Figure 4** in the next section.